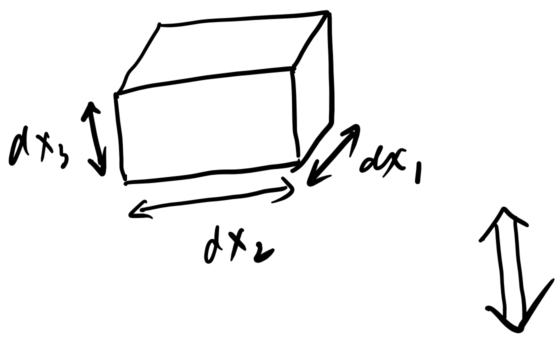
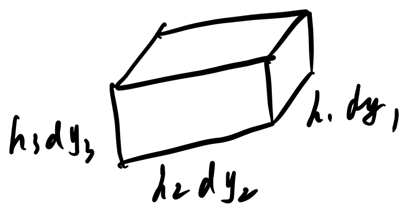
可以参看 Ernest S. Abers - Quantum mechanics

505 (A·3)

来梳理一下





在讲曲星之前
先讲 metric

h_i : scaling factor

$$(ds)^2 = (dx_1)^2 + (dx_2)^2 + (dx_3)^2$$

$$= g_{ij}\, dy_i\, dy_j$$

$$= \cdots$$

$$= (h_1 dy_1)^2 + (h_2 dy_2)^2 + (h_3 dy_3)^2$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial y_1} dy_1 + \frac{\partial \vec{r}}{\partial y_2} dy_2 + \frac{\partial \vec{r}}{\partial y_3} dy_3$$

$$= h_1 dy_1\, \hat{e}_1 + h_2 dy_2\, \hat{e}_2 + h_3 dy_3\, \hat{e}_3$$

$\hat{e}_i$: basis

$(h_1 dy_1 , h_2 dy_2 , h_3 dy_3)$

$\nabla\phi = ?$ （ϕ 增加最大的方向）

$$d\phi = \nabla\phi \cdot d\vec{r}$$

$$= \frac{\partial\phi}{\partial y_1} dy_1 + \frac{\partial\phi}{\partial y_2} dy_2 + \frac{\partial\phi}{\partial y_3} dy_3$$

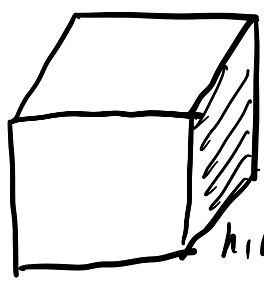
$$= \left(\frac{\partial\phi}{h_1 \partial y_1}, \frac{\partial\phi}{h_2 \partial y_2}, \frac{\partial\phi}{h_3 \partial y_3}\right)(h_1 dy_1, h_2 dy_2, h_3 dy_3)$$

$$\nabla = \left(\frac{\partial}{h_1 \partial y_1}, \frac{\partial}{h_2 \partial y_2}, \frac{\partial}{h_3 \partial y_3}\right)$$

: covariant vector

$d\vec{r}$: contravariant vector

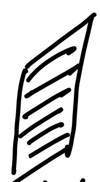
$\boxed{\text{divergence ?}}$



$h_3 dy_3$ $h_1 dy_1$ $h_2 dy_2$

$$(\nabla\cdot\vec{v})\, \underset{\displaystyle (h_1 h_2 h_3\, dy_1\, dy_2\, dy_3)}{\underset{\shortparallel}{d\tau}}$$

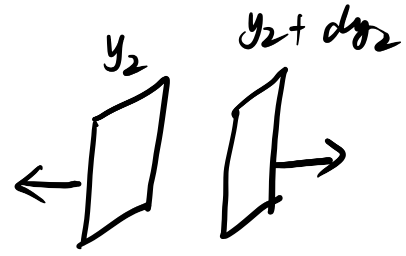
flux through the face



$h_3 dy_3$ $h_1 dy_1$

$$V_2\, h_1\, dy_1\, h_3\, dy_3$$

$$= V_2\, h_1\, h_3\, dy_1\, dy_3$$



两个对着的面

$$\left(\frac{\partial (v_2 h_1 h_3)}{\partial y_2} \right) dy_1\, dy_2\, dy_3$$

另外两 pair 面

$$\frac{\partial}{\partial y_1} (h_2 h_3 v_1) \, dy_1\, dy_2\, dy_3$$

$$\frac{\partial}{\partial y_3} (h_1 h_2 v_3) \, dy_1\, dy_2\, dy_3$$

total flux :

$$\vec{v} \cdot d\vec{a} = \left[\frac{\partial}{\partial y_1} (h_2 h_3 v_1) + \frac{\partial}{\partial y_2} (h_1 h_3 v_2) + \frac{\partial}{\partial y_3} (h_1 h_2 v_3) \right] dy_1\, dy_2\, dy_3$$

(Gauss theorem)

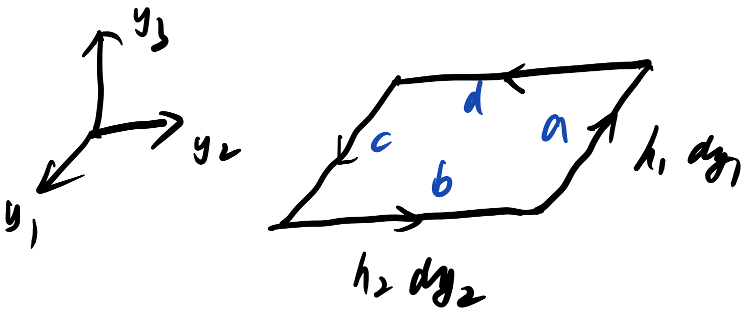
$$\nabla \cdot \vec{v} = \lim_{d\tau \to 0} \frac{\int \vec{v} \cdot d\vec{a}}{\int d\tau}$$

flux : surface integral

$$= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial y_1} (h_2 h_3 v_1) + \frac{\partial}{\partial y_2} (h_1 h_3 v_2) + \frac{\partial}{\partial y_3} (h_1 h_2 v_3) \right]$$

volume integral

$\boxed{\text{Curl ?}}$



(loop 如图)

$y_1 - y_2$ plane

$$(\nabla \times \vec{v}) \cdot d\vec{a} = \vec{v} \cdot d\vec{l}$$

a & c

$$- v_1 h_1 dy_1 \Big|_{y_2 + dy_2}$$

$$+ v_1 h_1 dy_1 \Big|_{y_2}$$

$$= - \frac{\partial (v_1 h_1)}{\partial y_2} dy_1 dy_2$$

b & d :

$$- v_2 h_2 dy_2 \Big|_{y_1}$$

$$+ v_2 h_2 dy_2 \Big|_{y_1 + dy_1}$$

$$= \frac{\partial (v_2 h_2)}{\partial y_1} dy_1 dy_2$$

$$\boxed{(\nabla \times \vec{v})_z = \frac{\vec{v} \cdot d\vec{l}}{da}}$$

$$da = h_1 h_2 \, dy_1 \, dy_2$$

$$= \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial y_1} (h_2 v_2) - \frac{\partial}{y_2} (h_1 v_1) \right]$$

surface integral

" line integral "

$$\nabla \times \vec{v} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \vec{e}_1 h_1 & \vec{e}_2 h_2 & \vec{e}_3 h_3 \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}$$

∇ :

the component of $\nabla \psi (y_1, y_2, y_3)$ in the direction normal to the family of surfaces $y_1 = \text{const}$

$$\nabla\psi\big|_1 = \frac{\partial\psi}{\partial s_1} = \frac{\partial\psi}{h_1\,\partial y_1}$$

$$ds_i = h_i\, dy_i$$

$$y_1 = r, \qquad y_2 = \theta, \qquad y_3 = \varphi$$

$$\nabla = \left(\frac{\partial}{\partial r},\ \frac{\partial}{r\,\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

match $(\vec{e}_r,\ \vec{e}_\theta\ \ \vec{e}_\varphi)$

等值面 (contour) 的法向

(增加最快的方向)

Check $\nabla\cdot\nabla\psi(y_1, y_2, y_3)$

注意：不能套用

$$\nabla\cdot\nabla \neq \left(\frac{\partial}{\partial r},\ \frac{\partial}{r\,\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\frac{\partial}{\partial r},\ \frac{\partial}{r\,\partial\theta},\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

∇ 定义从直角坐标系开始，用坐标系间的对应关系来推导 ∇, ∇^2 …
在一般性坐标系的表达式

$\nabla \cdot \underline{\nabla \psi}$ 将之视为矢量

$$\nabla \cdot \vec{v} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial y_1}(h_2 h_3 V_1) + \frac{\partial}{\partial y_2}(h_1 h_3 V_2) + \frac{\partial}{\partial y_3}(h_1 h_2 V_3)\right]$$

$$\nabla \psi = \left(\frac{\partial \psi}{h_1 \partial y_1}, \frac{\partial \psi}{h_2 \partial y_2}, \frac{\partial \psi}{h_3 \partial y_3}\right)$$

$$\nabla^2 \psi = \nabla \cdot \nabla \psi =$$

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial y_1}\left(h_2 h_3 \frac{\partial \psi}{h_1 \partial y_1}\right), \frac{\partial}{\partial y_2}\left(h_1 h_3 \frac{\partial \psi}{h_2 \partial y_2}\right), \frac{\partial}{\partial y_3}\left(h_1 h_2 \frac{\partial \psi}{h_3 \partial y_3}\right)\right)$$

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial y_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial y_1}\right) + \frac{\partial}{\partial y_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial y_2}\right) + \frac{\partial}{\partial y_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial y_3}\right)\right)$$

对于球坐标系

$h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

$y_1 = r$, $y_2 = \theta$, $y_3 = \varphi$

$\downarrow$

$$\nabla^2 = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right)\right)$$

$$\therefore \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$