antisymmetric tensor (levi-civita symbol)

2d

$$\varepsilon_{ij} \begin{cases} +1 & \text{if } (i,j) = (1,2) \\ -1 & \text{if } (i,j) = (2,1) \\ 0 & \text{if } i = j \end{cases}$$

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

3d $(1,2,3) \xrightarrow{\text{3 permutations}} (3,2,1)$

cyclic permutation of $(1,2,3)$

$$\varepsilon_{ijk} = \begin{cases} +1 & (1,2,3) \text{ or even permutations} \\ -1 & (3,2,1) \text{ or odd permutations} \\ 0 & \text{if two indices equal} \end{cases}$$

$$\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$$\varepsilon_{ijk}\,\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$$

Einstein notation, the duplication of i index implies the sum on i

$$\vec{a} \times \vec{b} = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \sum_{ijk} \varepsilon_{ijk}\, e_i\, a_j\, b_k$$

$$\nabla \times \nabla \times \vec{v}$$

$$(\nabla \times \vec{v})_i = \varepsilon_{ijk}\, \partial_j\, v_k$$

$$[\nabla \times (\nabla \times \vec{v})]_i$$

$$= \varepsilon_{ijk}\, \partial_j\, (\nabla \times \vec{v})_k$$

$$= \varepsilon_{ijk}\, \partial_j\, \varepsilon_{klm}\, \partial_l\, v_m$$

$$= \underline{\varepsilon_{ijk}\, \varepsilon_{klm}}\, \partial_j\, \partial_l\, v_m$$

$$\parallel$$

$$\sum_k \underline{\varepsilon_{ijk}\, \varepsilon_{lmk}}$$

$$= \underline{\varepsilon_{ij1}\, \varepsilon_{lm1}} + \varepsilon_{ij2}\, \varepsilon_{lm2} + \varepsilon_{ij3}\, \varepsilon_{lm3}$$

when 1st term $\neq 0$，其余2项必为0

因为 i, j 只能取 2 or 3

$\therefore \quad \varepsilon_{ijk}\,\varepsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

源自 $\underline{\varepsilon_{ijk}}$ 交换指标出负号

$\Downarrow$

$$(\nabla \times \nabla \times \vec{v})_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j\,\partial_l\,V_m$$

$$= \partial_j\,\partial_i\,V_j - \partial_j\,\partial_j\,V_i$$

$$= \partial_i\,\partial_j\,V_j - \partial_j\,\partial_j\,V_i$$

除 i 指标，其它都是 Einstein convention 求和

$\downarrow$

$$= \nabla(\nabla \cdot \vec{v}) - \nabla^2 \vec{v}$$

pit falls :

(隐患, 陷阱)

$$(\nabla\psi) \times (\nabla\phi) \; ?$$

two scalar functions ψ, ϕ

$$(\vec{A}\psi) \times (\vec{A}\phi) = 0$$

but

$(\nabla\psi) \times (\nabla\phi) \neq 0$ (generally)

ψ, ϕ is scalar field

∇ is a operator, the vector depends on the function behind

here, the two operators ∇ are not equal !

$\nabla \times \nabla\psi = 0$ holds for any scalar field

both ∇'s operate on the same function

两个矢量如何变成一个矢量

$a_i b_j$ — rank 2 tensor

如果直接 contraction $a_i b_i \to$ scalar

唯一方式 →

$a_i b_j \varepsilon_{klm} \quad \delta_{il} \delta_{jm}$

$1+1+3 \quad -2 \quad -2$

利用外积变成一个矢量

but $\varepsilon'_{\alpha\beta\gamma} = \det A \; a_{\alpha i} a_{\beta j} a_{\gamma k} \varepsilon_{ijk}$

$\det A = \pm 1$, make $\varepsilon_{ijk} \to$ pseudo tensor

so

outer product

$\times$	$\vec{v}$	$\vec{v}_p$
$\vec{v}$	$\vec{v}_p$	$\vec{v}$
$\vec{v}_p$	$\vec{v}$	$\vec{v}_p$

two real vectors $\Downarrow$ pseudo vector

due to $\det A$ once (from ε_{ijk})

$\vec{V} \times \vec{V_p}$

(real) (pseudo)

$\underline{\det A}$ twice !

$\longrightarrow$ real vector

then the volume is a real scalar ?

No !

$$\underbrace{(\vec{v_1} \times \vec{v_2})} \cdot \vec{v_3}$$

$\downarrow$

pseudo vector $\cdot$ real vector

$\Downarrow$

pseudo scalar