如何定义距离

distance（标量）

构成 vector（displacement）

如何变成 scalar

the simplest way is to define

（直角坐标系）

$d\vec{r} \cdot d\vec{r}$

$(dr)^2 = d\vec{r} \cdot d\vec{r}$ （Riemann）

$= (dx_i \vec{e}_i) \cdot (dx_j \vec{e}_j)$

$= \left(\frac{\partial \vec{r}}{\partial x_i} dx_i \right) \cdot \left(\frac{\partial \vec{r}}{\partial x_j} dx_j \right)$

$= (\vec{e}_i \cdot \vec{e}_j) dx_i dx_j$

$= \left(\frac{\partial \vec{r}}{\partial x_i} \cdot \frac{\partial \vec{r}}{\partial x_j} \right) dx_i dx_j$

$= g_{ij} dx_i dx_j$

$\frac{\partial \vec{r}}{\partial x} = \vec{e}_x \qquad \frac{\partial \vec{r}}{\partial y} = \vec{e}_y , \quad \frac{\partial \vec{r}}{\partial z} = \vec{e}_z$

g_{ij} : rank-2 tensor

come from the definition of distance

$$g_{ij} = \frac{\partial \vec{r}}{\partial x_i} \cdot \frac{\partial \vec{r}}{\partial x_j} \quad \Rightarrow \quad g_{ij} = g_{ji}$$

$$= \sum_a \frac{\partial r_a}{\partial x_i} \frac{\partial r_a}{\partial x_j}$$

(due to inner product)

E.g. $\vec{r} = \vec{r}(x, y, z)$

basis : $x_1 = r$, $x_2 = \theta$, $x_3 = \varphi$

$$g_{11} = 1$$
$$g_{22} = r^2$$
$$g_{33} = r^2 \sin^2\theta$$

$$g_{22} = \frac{\partial x}{\partial \theta} \frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta} \frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial \theta} \frac{\partial z}{\partial \theta}$$

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r\cos\theta \end{cases}$$

$$g = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2 \sin^2\theta \end{pmatrix}$$

$$ds^2 = g_{ij}\, dx_i\, dx_j$$

e.g. $g_{ij} = 0$, when $i \neq j$

we write $g_{ii} = h_i^2$

then $(dr)^2 = \sum_i (h_i \, dx_i)^2$

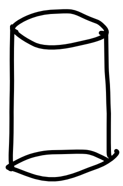
$$dr_i = h_i \, dx_i$$

for spherical coordinates

$$h_r = 1 \qquad h_\theta = r \qquad h_\varphi = r \sin\theta$$

$$dV = h_r h_\theta h_\varphi \, dr \, d\theta \, d\varphi$$

Q： 推导一下柱坐标、



$$\begin{matrix} q_1 & q_2 & q_3 \end{matrix}$$
$$(\rho, \theta, z)$$

$$x = \rho \cos\theta$$
$$y = \rho \sin\theta$$
$$z = z$$

$$g_{ij} = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}$$

$$g_{11} = \frac{\partial \begin{pmatrix} \rho \cos\theta \\ \rho \sin\theta \\ z \end{pmatrix}}{\partial \rho} \cdot \frac{\partial \begin{pmatrix} \rho \cos\theta \\ \rho \sin\theta \\ z \end{pmatrix}}{\partial \rho}$$

$$= \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

$$= 1$$

$$g_{22} = \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\rho\sin\theta \\ \rho\cos\theta \\ 0 \end{pmatrix}$$

$$= \rho^2$$

$$g_{33} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= 1$$

$$g = \begin{pmatrix} 1 & & \\ & \rho^2 & \\ & & 1 \end{pmatrix}$$

$$h_\rho = 1 \quad , \quad h_\theta = \rho \, . \quad h_z = 1$$

$$d\vec{r} = \hat{e}_\rho \, d\rho + \rho \, \hat{e}_\theta \, d\theta + \hat{e}_z \, dz$$

$$g_{ij} = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}$$

$\cancel{x_i \to q_i}$

$$dr_i = h_i \, dq_i$$

$$= h_i \vec{e}_i \cdot h_j \vec{e}_j$$

$$\frac{\partial \vec{r}}{\partial q_i} = h_i \hat{e}_i$$

$$h_i = (g_{ii})^{1/2}$$

$$(dr)^2 = g_{ij} \, dx_i \, dx_j$$

$$= \sum_{ij} (h_i \, dx_i) \vec{e}_i \cdot (h_j \, dx_j) \vec{e}_j$$

$$d\vec{r} = h_1 \, dq_1 \, \hat{e}_1 + h_2 \, dq_2 \, \hat{e}_2 + h_3 \, dq_3 \, \hat{e}_3$$

spherical polar coordinates

cylindrical coordinates

$$\vec{r} \neq r \hat{e}_r + \theta \hat{e}_\theta + \varphi \hat{e}_\varphi$$

$\hat{e}_i$ is space dependent

$$\vec{r} \neq \rho \hat{e}_\rho + \theta \hat{e}_\theta + z \hat{e}_z$$

Dealing with an arbitrary curvilinear system, with coordinates labeled (q_1, q_2, q_3), we consider how changes in the q_i are related to changes in the Cartesian coordinates.

$$x(q_1, q_2, q_3)$$

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$(dr)^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$(dx)^2 = \sum_{ij} \frac{\partial x}{\partial q_i} \frac{\partial x}{\partial q_j} dq_i \, dq_j$$

$$(dr)^2 = \sum_{ij} g_{ij} \, dq_i \, dq_j \qquad (*)$$

$$g_{ij}(q_1, q_2, q_3) = \sum_{ij} \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}$$

$$= \frac{\partial x}{\partial q_i} \frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i} \frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i} \frac{\partial z}{\partial q_j}$$

spaces with a measure of distance given by $(*)$ are called metric or Riemannian

方向导数 (directional derivative)

in 2d: $\left.\frac{\partial f}{\partial l}\right|_{(x_0,y_0)}$ or $\left.\frac{\partial f}{\partial \vec{n}}\right|_{(x_0,y_0)}$

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial l} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial l}$$

$$= \nabla f \cdot \vec{e}_l$$

$$\vec{e}_l = \left(\frac{\partial x}{\partial l}, \frac{\partial y}{\partial l}\right)$$

$\vec{l}$ 的单位矢量 : $\vec{e}_l$ $\qquad \vec{l} = l\,\vec{e}_l$

$$\vec{e}_l = \frac{\vec{l}}{|\vec{l}|}$$

in 3d:

$$\frac{\partial f}{\partial \hat{l}} = \lim_{l\to 0} \frac{f(x+l\cos\alpha,\ y+l\cos\beta,\ z+l\cos\gamma) - f(x,y,z)}{l}\,\hat{e}_l$$

$$= \lim_{l\to 0} \frac{\frac{\partial f}{\partial x} l\cos\alpha + \frac{\partial f}{\partial y} l\cos\beta + \frac{\partial f}{\partial z} l\cos\gamma}{l}\,\hat{e}_l$$

$$= (\nabla f \cdot \vec{e}_l)\,\hat{e}_l$$

$\hat{e}_l = (\cos\alpha, \cos\beta, \cos\gamma)$

α, β, γ 为 $\hat{e}_l$ 与 x, y, z 方向的夹角。