$$E = \frac{1}{2}\dot{\mathbf{r}}^2 - k/r$$

Arnold

《经典力学的数学方法》

$$= \frac{1}{2}\dot{r}^2 + \frac{1}{2}(r\dot{\varphi})^2 - k/r$$

$$= \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{M^2}{r^2} - k/r$$

$$= \frac{1}{2}\dot{r}^2 + V(r)$$

$U(r) = -\frac{k}{r}$, $M = r^2\dot{\varphi}$，质量为1

$\dot{\varphi} = M/r^2$, $\dot{r} = \sqrt{2(E-V)}$

$$\frac{d\varphi}{dr} = \frac{M/r^2}{\sqrt{2(E-V)}}$$

$$\varphi = \int \frac{M/r^2 \, dr}{\sqrt{2(E-V(r))}} \qquad (*)$$

$$\varphi = \arccos \frac{\frac{M}{r} - \frac{k}{M}}{\sqrt{2E + \frac{k^2}{M^2}}}$$

$$\downarrow$$

$\frac{M^2}{k} = p$, $\sqrt{1 + \frac{2EM^2}{k^2}} = e$

$$\varphi = \arccos[((p/r) - 1)/e]$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$f(r) = \left(\frac{M}{r} - \frac{k}{M}\right) \Big/ \sqrt{2E + \frac{k^2}{m}}$$

$$(\arccos f(r))' = -\frac{1}{\sqrt{1-f(r)^2}} \, f'(r)$$

E, M conserved, k is a constant

$$\varphi = \arccos f(r)$$

$$\frac{d\varphi}{dr} = \frac{M/r^2}{\sqrt{2(E-V)}}$$

$$f(r)^2 = \left(\frac{M}{r} - \frac{k}{M}\right)^2 \Big/ \left(2E + \frac{k^2}{M^2}\right)$$

$$1 - f(r)^2 = \frac{2E + \cancel{\frac{k^2}{M^2}} - \frac{M^2}{r^2} - \cancel{\frac{k^2}{M^2}} + 2\frac{k}{r}}{2E + k^2/M^2}$$

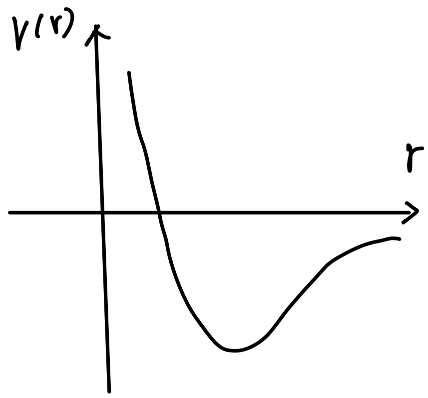
$$= \frac{2(E-V)}{2E + k^2/M^2}$$

$$(\arccos f(r))' = -\sqrt{\frac{2E + k^2/m^2}{2(E-V)}}\left(-\frac{m}{r^2}\right)\Big/\sqrt{2E+\frac{k^2}{m^2}}$$

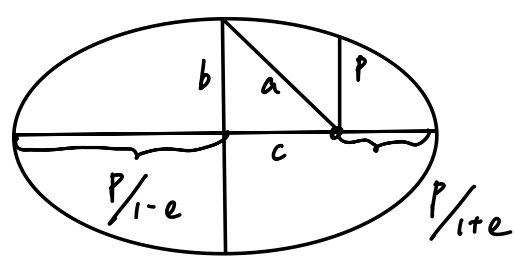
$$= \frac{1}{\sqrt{2(E-V)}}\left(\frac{m}{r^2}\right)$$

$$\sqrt{1+\frac{2Em^2}{k^2}} = e$$

$$\frac{d\varphi}{dr} = \arccos f(r)$$



$$\varphi = \arccos[((p/r)-1)/e]$$



有效势能

E<0时运动是有界的

如果以 $\frac{1}{r}$ 为变量看问题会简单很多

(from 赵SX)

$$d\varphi = \int_{r_0}^{r} \frac{M/r^2}{\sqrt{2\left(E+\frac{k}{r}-\frac{M^2}{2r^2}\right)}}\, dr$$

$$\frac{1}{r} \to u$$

$$d\varphi = -\int_{\frac{1}{r_0}}^{\frac{1}{r}} \frac{du}{\sqrt{c_0 + c_1 u - u^2}}$$

$$c_0 = \frac{2E}{M^2}, \quad c_1 = \frac{2k}{M^2}$$

$$d\varphi = \int_{\frac{1}{r}}^{\frac{1}{r_0}} \frac{du}{\sqrt{-\left(u-\frac{c_1}{2}\right)^2 + c_0 + \frac{c_1^2}{4}}} \qquad c_0 + \frac{c_1^2}{4} = c_2^2$$

$$\boxed{\begin{gathered} \varphi - \varphi_0 \\ 取\ \varphi_0 = \frac{\pi}{2} \\ r_0 = p = \frac{M^2}{k} \end{gathered}}$$

$$= \int_{\frac{1}{r}}^{\frac{1}{r_0}} \frac{du/c_2}{\sqrt{1-\left(\frac{u-\frac{c_1}{2}}{c_2}\right)^2}}$$

$$= \arcsin\left(\frac{u}{c_2}\right) \Bigg|_{\left(\frac{1}{r}-\frac{c_1}{2}\right)/c_2}^{\left(\frac{1}{r_0}-\frac{c_1}{2}\right)/c_2}$$

$(\Delta)\ \sin\left(\varphi - \frac{\pi}{2}\right) = \frac{1}{C_2}\left[\frac{1}{r_0} - \frac{1}{r}\right]$

$\cos\varphi = \frac{1}{C_2}\left[\frac{1}{r} - \frac{1}{r_0}\right]$

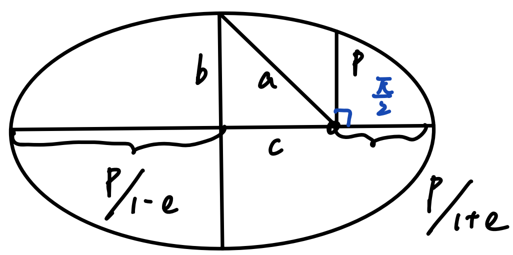
比较: $\frac{M^2}{K} = p$, $\sqrt{1 + \frac{2EM^2}{K^2}} = e$

$(\ast)\quad \varphi = \arccos[((p/r) - 1)/e]$

$C_2 = \sqrt{\frac{2E}{M^2} + \frac{K^2}{M^4}} = e\frac{K}{M^2}$

按照结论反推：

$r_0 = p$



$(\ast)\ 1 + e\cos\varphi = p/r$

$(\Delta)\quad e\cos\varphi = \frac{M^2}{K}\left(\frac{1}{r} - \frac{1}{p}\right)$

$= p/r - 1$

$1 + e\cos\varphi = p/r$

Lagrangian :

$$L = \frac{1}{2}\dot{\vec{r}}^{2} - k/r$$

$$= \frac{1}{2}\dot{r}^{2} + \frac{1}{2}r^{2}\dot{\varphi}^{2} - k/r$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

equation of motion

$q = r \Rightarrow$

$$\frac{d}{dt}(\dot{r}) - r\dot{\varphi}^{2} + k/r^{2} = 0$$

$$\ddot{r} = r\dot{\varphi}^{2} - k/r^{2}$$

$q = \varphi \Rightarrow$

$$\frac{d}{dt}(r^{2}\dot{\varphi}) = 0$$

$$r^{2}\dot{\varphi} = \text{const}$$

$r \; r\dot{\varphi} = \text{const} \Rightarrow M$ (角动量守恒)

$$r^{2}\frac{d\varphi}{dt} = ds/dt$$