再谈 notation：

$$\vec{r} = (x, y, z)$$

设三维空间中的矢量，当参考原点选定，矢量的 head 位于坐标 (x, y, z) 处

而 $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$(x, y, z)$$

表示取分量 x, y, z 沿着 $\hat{x}$, $\hat{y}$, $\hat{z}$ 方向

$$\vec{F} = -\nabla U$$

$F_x = -\frac{\partial U}{\partial x}$, $F_y = -\frac{\partial U}{\partial y}$, $F_z = -\frac{\partial U}{\partial z}$

（对应分量相等）

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$$

due to $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$

$\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{z} = \hat{x}\cdot\hat{z} = 0$

review:

$$\nabla\varphi = |\nabla\varphi|\,\vec{e}_{\nabla\varphi}$$

$\vec{e}_{\nabla\varphi}$ 是 标量场 φ 增加最快的方向

因为 $\nabla\varphi = \left(\frac{\partial}{\partial x}\varphi, \frac{\partial}{\partial y}\varphi, \frac{\partial}{\partial z}\varphi\right)$

$$\vec{e}_x \frac{\partial}{\partial x}\varphi = \vec{e}_x \frac{\varphi(x+\Delta x, y, z) - \varphi(x, y, z)}{\Delta x}$$

Δx 默认方向是 $\vec{e}_x$ (正向)

$\Delta\varphi$ 为正，即为增加

这样朝 $\vec{e}_x$ 正向，增加方向

$\Delta\varphi$ 为负，结果为

$-\vec{e}_x$ 方向，也是 φ 增加方向

爬山：

$h(x+\Delta x, y+\Delta y) - h(x, y)$

$$\Delta h = \frac{\partial h}{\partial x}\Delta x + \frac{\partial h}{\partial y}\Delta y$$

$$\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right) \cdot (\Delta x, \Delta y)$$

向量点乘

$\Delta x^2 + \Delta y^2 = \Delta r^2$

步长，方向

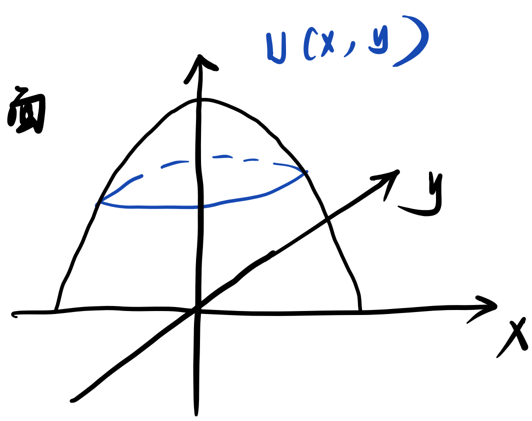
同 $\left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$

使得 Δh 最大

set $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ (in 2d)

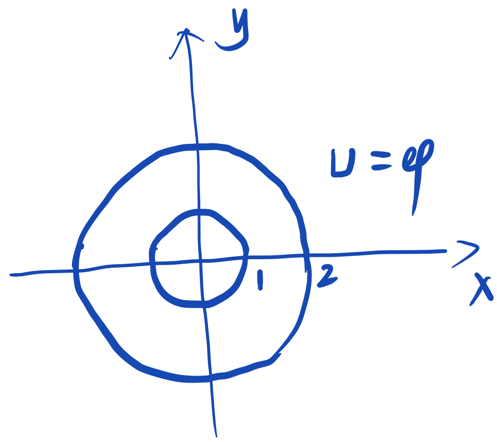
$U(x, y) = x^2 + y^2$

等高面



电场线

等高线



力：

∇U 的方向

$$U(B) - U(A) = -\int \vec{F} \cdot d\vec{r}$$

做功原因 $\quad dU = -\vec{F} \cdot d\vec{r} = -F_x\,dx - F_y\,dy$

$$F_x = -\frac{\partial U}{\partial x}$$

$$F_y = -\frac{\partial U}{\partial y}$$

$$\vec{F} = -\nabla U$$

保守力判据：$\quad \dfrac{\partial F_x}{\partial y} = \dfrac{\partial F_y}{\partial x}$

$$\frac{\partial^2 U}{\partial x\,\partial y} = \frac{\partial^2 U}{\partial y\,\partial x}$$

$U(x, y)$ is "smooth"

Conservation of momentum:

若 $\vec{F_e}=0$ (合外力为0)

$m_1\vec{a_1}=\vec{F_{12}}$ $\Rightarrow$ $m_1\frac{d\vec{v_1}}{dt}+m_2\frac{d\vec{v_2}}{dt}=0$

$m_2\vec{a_2}=\vec{F_{21}}$ $\quad d(m\vec{v_1}+m\vec{v_2})/dt=0$

$m_1\vec{a_1}+m_2\vec{a_2}=0$ $\quad \frac{d\vec{p}}{dt}=0$

$$\vec{R}=\frac{\sum_i m_i\vec{r_i}}{\sum_i m_i}$$

$(M=\sum_i m_i)$

$\vec{r_i}$的加权平均

(两体)

$\frac{d^2}{dt^2}(m\vec{r_1}+m_2\vec{r_2})=0$

(mass center) 定义

$\vec{p}=\sum_i m_i\vec{v_i}=M\frac{d\vec{R}}{dt}=M\vec{V}$ (质心速度)

$$\vec{F_e}=M\frac{d^2\vec{R}}{dt^2}$$

$m_1\vec{a_1}=\vec{F_{12}}+\vec{F_{e1}}$

$m_2\vec{a_2}=\vec{F_{21}}+\vec{F_{e2}}$

$m_1\vec{a_1}+m_2\vec{a_2}=\vec{F_{e1}}+\vec{F_{e2}}=\vec{F_e}$

$m_1\frac{d^2\vec{r_1}}{dt^2}+m_2\frac{d^2\vec{r_2}}{dt^2}=\vec{F_e}$

$\frac{d^2}{dt^2}(m_1\vec{r_1}+m_2\vec{r_2})=\vec{F_e}$ (质心加速度)

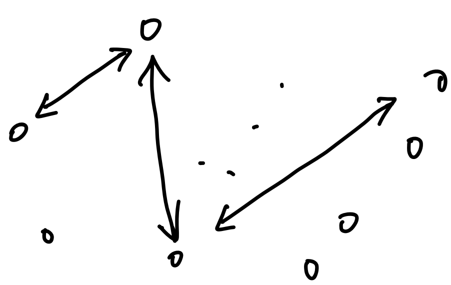
$\frac{d^2}{dt^2}(M\vec{R})=\vec{F_e}\Rightarrow\vec{F_e}=M\frac{d^2\vec{R}}{dt^2}$

action reaction

each pair of particles has corresponding components of mutual interaction that are equal in magnitude and opposite in direction

two-body interaction

three-body problem so long challenged human powers of analysis



(why the orbitals of sun, earth, moon are analytically determined?)

internal / external forces

$$m\vec{a} = \vec{F}_{in} + \vec{F}_{ex}$$

sum over all objects. $\vec{F}_{in}^{tot} = 0$

two-body case :

(动量守恒)

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \frac{m_1\vec{r}_1' + m_2\vec{r}_2'}{m_1+m_2}$$

($\vec{r}_1$, $\vec{r}_2$ 的加权平均)

$$\Delta\vec{r}_1 = \vec{r}_1 - \vec{R} \qquad \Delta\vec{r}_2 = \vec{r}_2 - \vec{R}$$

$$\frac{d\Delta\vec{r}_1}{dt} = \frac{d\vec{r}_1}{dt} - \frac{d\vec{R}}{dt}$$

$$\Delta\vec{v}_1 = \vec{v}_1 - \vec{v}_c$$

(质心速度 $\vec{v}_c$)

$$\frac{d\Delta\vec{r}_2}{dt} = \frac{d\vec{r}_2}{dt} - \frac{d\vec{R}}{dt}$$

$$\Delta\vec{v}_2 = \vec{v}_2 - \vec{v}_c$$

$$E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

$$= \frac{1}{2}m_1(\Delta\vec{v}_1+\vec{v}_c)^2 + \frac{1}{2}m_2(\Delta\vec{v}_2+\vec{v}_c)^2$$

$$= \frac{1}{2}m_1\Delta v_1^2 + \frac{1}{2}m_1v_c^2 + m_1\Delta\vec{v}_1\cdot\vec{v}$$

$$+ \frac{1}{2}m_2\Delta v_2^2 + \frac{1}{2}m_2v_c^2 + m_2\Delta\vec{v}_2\cdot\vec{v}$$

(相等撤另初为碰撞前)

$$(m_1\Delta\vec{v}_1 + m_2\Delta\vec{v}_2)\cdot\vec{v} = 0$$

(质心看系统动量)

$$\frac{d}{dt}\left[m_1\vec{r}_1 + m_2\vec{r}_2 - (m_1+m_2)\vec{R}\right] = 0$$

$$= \frac{1}{2} m_1 \Delta V_1^2 + \frac{1}{2} m_2 \Delta V_2^2 + \underbrace{\boxed{\frac{1}{2}(m_1+m_2) V_c^2}}_{\text{质心速度不变 const}}$$

能量损失最大，$\Delta V_1 = \Delta V_2 = 0$

相对于质心静止

完全非弹性碰撞，粘在一起

$\Delta \vec{r}_1 = \Delta \vec{r}_2 = 0$（质点对质心偏离）

也可以做减法：

$$\Delta E_k = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 - \frac{1}{2} m_1 V_1'^2 - \frac{1}{2} m_2 V_2'^2$$

$$= \frac{1}{2} m_1 (\Delta \vec{V}_1 + \vec{V}_c)^2 + \frac{1}{2} m_2 (\Delta \vec{V}_2 + \vec{V}_c)^2$$

$$- \frac{1}{2} m_1 (\Delta \vec{V}_1' + \vec{V}_c)^2 - \frac{1}{2} m_2 (\Delta \vec{V}_2' + \vec{V}_c)^2$$

$$= \frac{1}{2} m_1 [\Delta V_1^2 - \Delta V_1'^2 + 2 \vec{V}_c \cdot (\Delta \vec{V}_1 - \Delta \vec{V}_1')]$$

$$+ (1 \rightarrow 2)$$

$$= \frac{1}{2} m_1 \Delta V_1^2 + \frac{1}{2} m_2 \Delta V_2^2 - \underbrace{\frac{1}{2}(m_1 \Delta V_1'^2 + m_2 \Delta V_2'^2)}_{\text{初值，已知}}$$

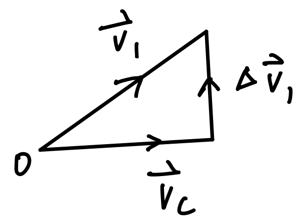
当 $\Delta V_1 = \Delta V_2 = 0$

末态相对于质心速度为 0

交叉项 $\vec{v}_c \cdot [\, m_1(\Delta\vec{v}_1 - \Delta\vec{v}_1') + m_2(\Delta\vec{v}_2 - \Delta\vec{v}_2') \,]$

$= \vec{v}_c \cdot \Delta\vec{p}$

$\Delta\vec{p} = 0$: 动量守恒



$\Delta\vec{v}_1 = \vec{v}_1 - \vec{v}_c$

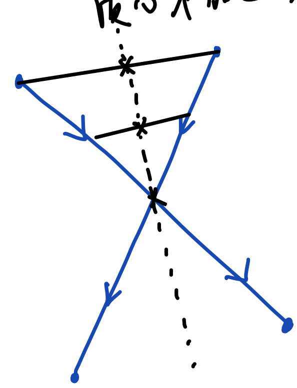
$\Delta\vec{v}_1' = \vec{v}_1' - \vec{v}_c$

$\Delta\vec{v}_1 - \Delta\vec{v}_1' = \vec{v}_1 - \vec{v}_1'$

$m_1(\Delta\vec{v}_1 - \Delta\vec{v}_1') = m_1(\vec{v}_1 - \vec{v}_1') = \Delta\vec{p}_1$

m_1 的动量变化

质心不加速，匀速直线运动



$$\frac{d^2\vec{R}}{dt^2} = 0 = \vec{F}_e$$