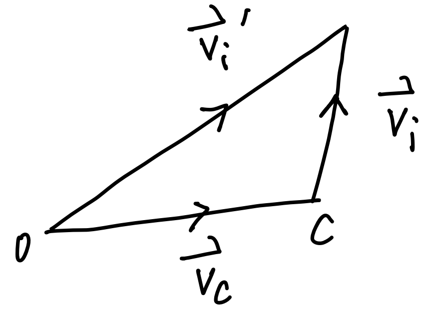
柯尼希定理

(König)



$$\vec{v}_i' = \vec{v}_c + \vec{v}_i$$

Kinetic energy referring to O:

$$\sum_i \frac{1}{2} m_i (\vec{v}_i' \cdot \vec{v}_i') = \sum_i \frac{1}{2} m_i (\vec{v}_c + \vec{v}_i) \cdot (\vec{v}_c + \vec{v}_i)$$

$$= \sum_i \frac{1}{2} m_i v_c^2 + \sum_i \frac{1}{2} m_i v_i^2$$

the momentum refer to the mass center reference

$$+ \sum_i \boxed{(m_i \vec{v}_i) \cdot \vec{v}_c}$$

$$(0 = \vec{p}_c) \cdot \vec{v}_c$$

$(\vec{v}_i = \vec{\omega} \times \vec{r}_i)$

$$= \frac{1}{2} m v_c^2 + \frac{1}{2} \sum_i m_i v_i^2$$

($E_{k,in}$) ? $= E_k^{cm}$

(质心的动能)

$$\| \quad \sum_i m_i r_i^2 \omega^2$$

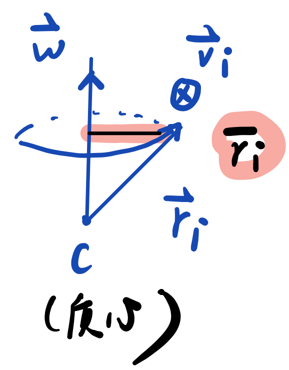
Q：相对质心只有转动吗？

$$= \underbrace{\frac{1}{2} m v_c^2}_{\text{质心的平动动能}} + \underbrace{\frac{1}{2} I_c \omega^2}_{\text{绕质心的转动动能}}$$

绕质心的转动动能：　$\vec{r}_i$ vs $\bar{r}_i$

$\vec{r}_i$：　质心到质元 m_i 的连线矢量

$\bar{r}_i$ 的长度：过质心的转轴与 m_i 所在的转动平面（$\perp \vec{\omega}$ 的平面）的交点与 m_i 所在质元的距离）



$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$
$$= \omega r_i \sin\theta$$
$$= \omega \bar{r}_i \hat{v}_i$$

$$\frac{1}{2} \sum_i m_i v_i^2$$
$$= \frac{1}{2} \sum_i m_i \bar{r}_i^2 \omega^2$$
$$= \frac{1}{2} \underset{\downarrow}{I_c} \omega^2$$

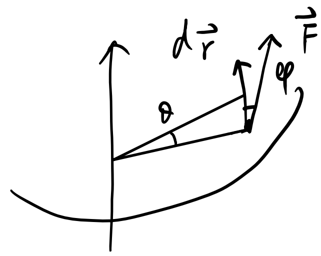
（系统相对于穿过质心的转轴的转动惯量）

转动动能表达式：

handwaving ：（比划一下）

$$\frac{1}{2}mv^2 \xrightarrow{v=\omega r} \frac{1}{2}I\omega^2$$

from calculus



做功微元： $dW = \vec{F}\cdot d\vec{r}$

$= F\,|d\vec{r}|\cos\varphi$

$= \underline{F\cos\varphi}\, R\, d\theta = \underline{F_{tan}}\, R\, d\theta$

$d\vec{r}$ 方向

$\vec{\tau} = \vec{R}\times\vec{F}$

$\boxed{= \tau\, d\theta}$

$= I\alpha\, d\theta$

$\vec{\tau} = \frac{d\vec{L}}{dt}$

$= I\frac{d\omega}{dt}\, d\theta$

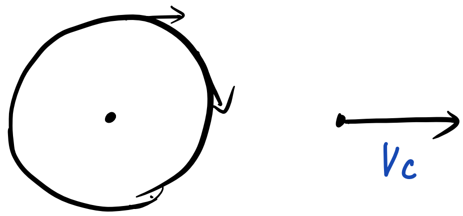
$\underline{\vec{\tau}\, dt} = d\vec{L}$

冲量矩

$= I\,\omega\, d\omega$

$= I\, d\omega^2$

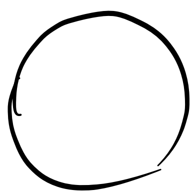
two motions : (can be independent)



the center of mass translation

1° no friction : $\omega \neq 0$, $V_c = 0$

(rotation)



2° brake : $V_c \neq 0$, $\omega = 0$

rolling without slipping

骑自行车 : $\frac{1}{2} m V_c^2 + \underline{\frac{1}{2} I_c \omega^2}$

$$\frac{1}{2} \frac{mR^2}{2} \frac{v^2}{R^2}$$

$$= \frac{1}{2} m V_c^2 + \frac{1}{4} m v^2$$

$$v = V_c$$

转一圈，平动 $2\pi R$

$$E_k = \frac{1}{2} m v_c^2 + \frac{1}{4} m v_c^2$$

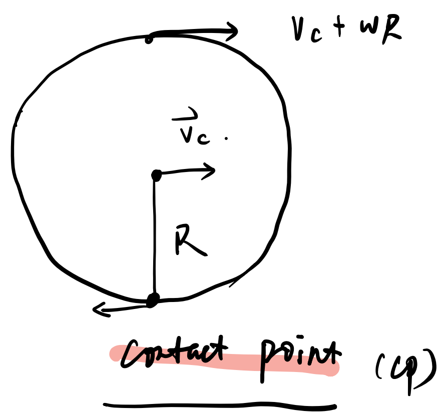
$$= \frac{1}{2}\left(mR^2 + \frac{1}{2} mR^2 \right) \omega^2 = \frac{1}{2}\left(m + \frac{1}{2} m \right) v^2$$

parallel axis theorem

$E_k = \frac{1}{2} I_{cp} \omega^2$

视为相对
接触点的转
动动能

(effective
full rotation)



$I_c = \frac{1}{2} mR^2$

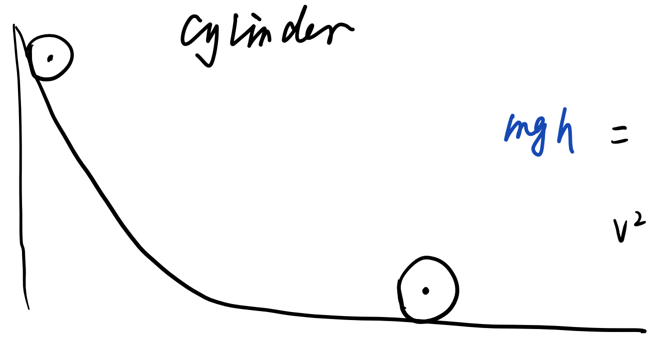
$I_{cp} = I_c + mR^2$

it is a fixed point

$V_{cp} = V_c - \omega R = 0$ (reference point: $V_{cp} = 0$)

Eg. 1

cylinder



$mgh = \frac{1}{2}\left(m + \frac{1}{2} m \right) v^2$

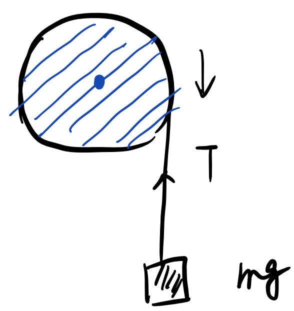
$v^2 = \frac{4}{3} gh \neq 2gh$

Eg.2

求 a ?

滑轮 (pulley)

translation　　rotation



$R\alpha = a$

(one string)

几何约束

受力分析：

$mg - T = ma$. (平动)

$R\alpha = a$

滑轮质量

$\tau = TR = I\alpha = \frac{1}{2} M R^2 \alpha = \frac{1}{2} M R a$.

(转动)

(tangent direction)

$T = \frac{1}{2} M a$

$$mg = \left(\frac{1}{2}M + m\right)a.$$

$$a = \frac{mg}{\left(\frac{1}{2}M + m\right)}$$

check it !

from Energy conservation :

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)w^2$$

$$= \frac{1}{2}mv^2 + \left(\frac{1}{4}M\boxed{R^2w^2} = \frac{1}{4}Mv^2\right)$$

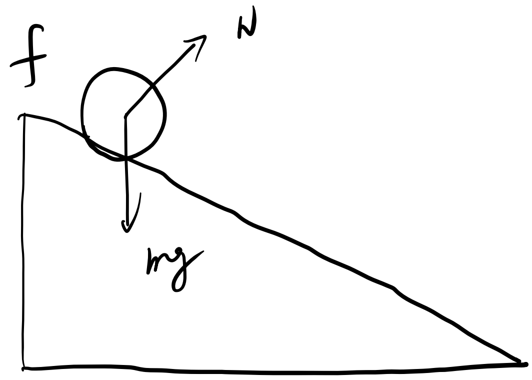
$$2ah = v^2 \Rightarrow h = \frac{v^2}{2a}$$

$$mg\,\cancel{v^2}/2a = \frac{1}{2}m\cancel{v^2} + \frac{1}{4}M\cancel{v^2}.$$

$$a = \frac{mg}{2\left(\frac{1}{2}m + \frac{1}{4}M\right)} = \frac{mg}{m + \frac{1}{2}M}$$

Eg. 3

CM :



$$\begin{cases} R\alpha = a & ① \quad (\text{geometry}) \\ mg\sin\theta - f = ma & ② \quad (\text{translation}) \\ fR = I\alpha & ③ \quad (\text{rotation}) \end{cases}$$

torque

check 能量

守恒

$$mg\sin\theta R - fR = maR$$

$$fR = I\alpha$$

$$mg\sin\theta R = maR + I\alpha$$

$$= maR + \frac{1}{2}mR^2\alpha$$

$$g\sin\theta = a + \frac{1}{2}R\alpha$$

$$= \frac{3}{2}a$$

$$\Rightarrow a = 2g\sin\theta/3$$

总结：

在这3个例子中，都有2/3的影子

Eg. 2，对 $M = m$

$$a = \frac{2}{3} g$$

Eg. 1 $v^2 = 2gh \times \frac{2}{3}$

均匀质量分布的圆盘（圆柱）