I. Each planet moves around the sun in an ellipse, with the sun at one focus.

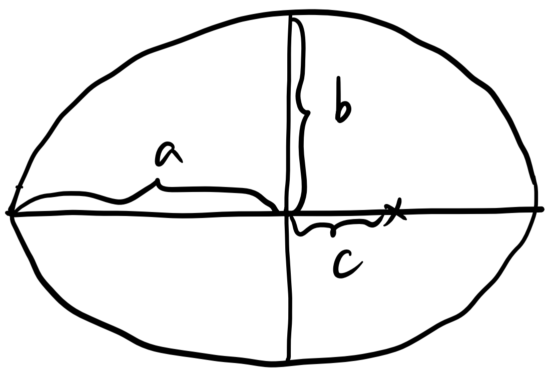
II. The radius vector from the sun to the planet sweeps out equal areas in equal intervals of time.

III. The squares of the periods of any two planets are proportional to the cubes of the semimajor axes of their respective orbits: $T \propto a^{3/2}$.

1° 椭圆轨道 fixed star 恒星

闭合，平面 planet 行星

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a^2 - b^2 = c^2$$



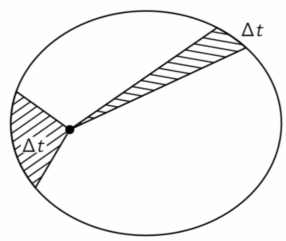


Fig. 7-2. Kepler's law of areas.

2° $\frac{ds}{dt} = \text{const}$

$\vec{L}$ 守恒

$\vec{\tau} = 0 \Rightarrow \vec{r} \times \vec{F} = 0$

$\vec{F} \propto \hat{r} \rightarrow$

(Feynman)

$2ds = \vec{r} \times \dot{\vec{r}}\, dt$

$$2\frac{d^2s}{dt^2} = \dot{\vec{r}} \times \dot{\vec{r}}\ (=0) + \vec{r} \times \ddot{\vec{r}} = \vec{r} \times \vec{F}/m = 0$$

3° $T^2/a^3 = \text{const}$

$$E = E_k + U_g$$

$$= \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$= \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2 - \frac{GMm}{r}$$

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= v_r\hat{r} + \underbrace{r\frac{d\theta}{dt}}_{wr = v_\theta}\hat{\theta}$$

(径向)　　(角向速度)

$$L = Iw = mr^2 w$$

(星体视为质点，质量 m 在质心处)

$$\frac{1}{2}mv_\theta^2 = \frac{1}{2}m w^2 r^2$$

$$= \frac{1}{2}m\left(\frac{L}{mr^2}\right)^2 r^2$$

$$= \frac{1}{2}\frac{L^2}{mr^2}$$

(※1)

$$E = \frac{1}{2} m v_r^2 + \tilde{U}(r)$$

$$\tilde{U}(r) = \frac{1}{2}\frac{L^2}{mr^2} - \frac{GMm}{r}$$

二维 → 一维

变量为 r

新的势能函数

$$v_r^2 = \frac{2E}{m} - \frac{2\tilde{U}}{m}$$

Equation of motion:

$$\frac{dr}{dt} = \sqrt{\frac{2(E-\tilde{U})}{m}}$$

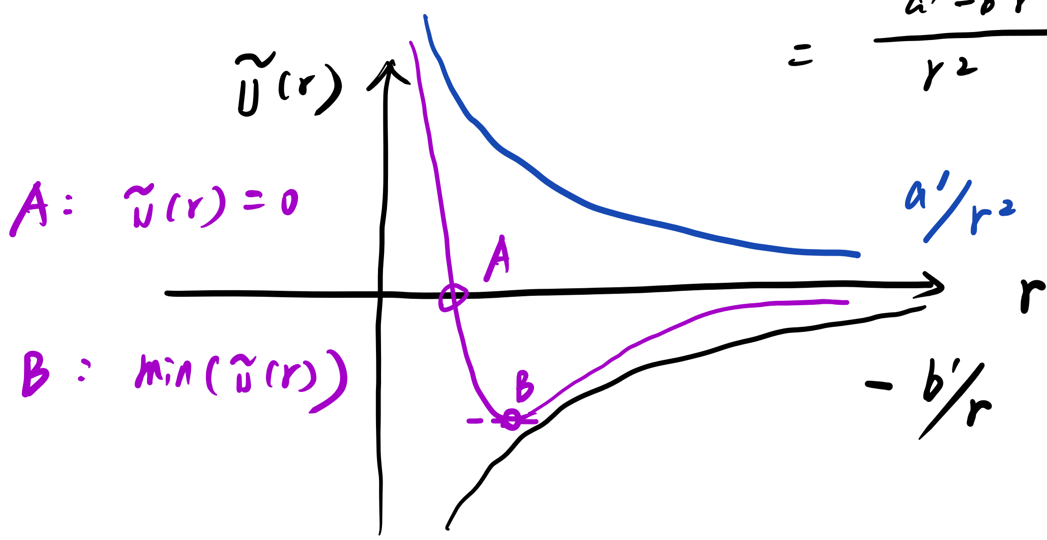
$$dt = \sqrt{\frac{m}{2(E-\tilde{U})}}\, dr$$

$$\tilde{U}(r) = \frac{1}{2}\frac{L^2}{mr^2} - \frac{GMm}{r}$$

$$\left(= \frac{1}{2} I \omega^2 \right)$$

$$= \frac{a'}{r^2} - \frac{b'}{r} \qquad b'>0$$

$$= \frac{a'-b'r}{r^2} \xrightarrow{r\to\infty} 0^-$$



$$\tilde{U}(r) = \frac{L^2}{2m}\left(\frac{1}{r} - \tilde{a}\right)^2 + \tilde{b}$$

$$-2\tilde{a}\,\frac{L^2}{2m} = -GMm$$

$$\tilde{a} = \frac{GMm^2}{L^2}$$

$$\tilde{b} = -\frac{L^2}{2m}\tilde{a}^2$$

$$= -\frac{\cancel{L^2}}{2m}\,\frac{G^2M^2m^4}{\cancel{L^4}\,L^2}$$

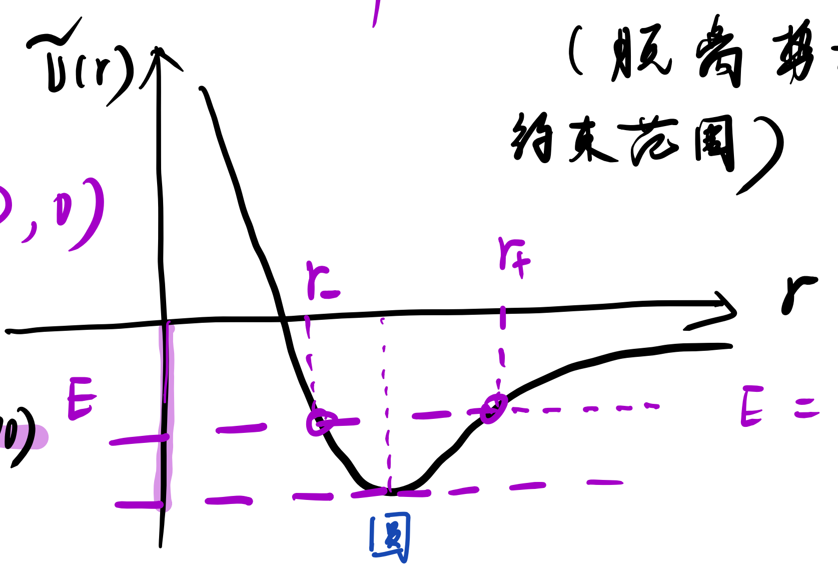
$$= -\frac{G^2M^2m^3}{2L^2}$$

$$\tilde{U}(r) \xrightarrow{r\to\infty} 0^-$$

$E < 0$，否则会 go to infinity

（脱离势场的约束范围）

$\therefore E \in (\tilde{U}(r_B), 0)$

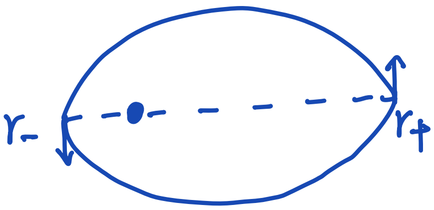


$V_r = 0$

（径向动能加）

$E = \tilde{U}(r)$

$r_+ = ?$ ， $r_- = ?$



远日点 近日点

($v_r = 0$)

$$E = \frac{1}{2}\frac{L^2}{mr^2} - \frac{GMm}{r}$$

$$r^2 + \frac{GMm}{E}r - \frac{L^2}{2Em} = 0$$

$$r_+ + r_- = -\frac{GMm}{E} = 2a$$

$$a = -\frac{GMm}{2E} \xrightarrow{\text{能量守恒}} \text{const}$$

与角动量无关

能量相同，a相同

$$r_\pm = \frac{-\frac{GMm}{E} \pm \sqrt{\left(\frac{GMm}{E}\right)^2 + \frac{2L^2}{Em}}}{2}$$

$$r_+ - r_- = 2c \implies$$

$$c = \sqrt{\underbrace{\left(\frac{GMm}{2E}\right)^2}_{a^2} + \underbrace{\frac{L^2}{2Em}}_{-b^2}} \xrightarrow{\text{角动量也守恒}} \text{const}$$

角动量改变偏心率

set $b \equiv \sqrt{-\frac{L^2}{2mE}}$ ($E<0$)

椭圆假定，check done !

if $c = 0$, 圆轨道

$$\left(\frac{GMm}{2E}\right)^2 = -\frac{L^2}{2Em}$$

$$E = -\frac{G^2M^2m^3}{2L^2}$$

$L_{max} = GMm\sqrt{\frac{m}{-2E}}$

$L \in (0, L_{max})$

for circular orbit, $L = rmv$

$L = 0$: 纯径向运动

$$E = -\frac{G^2M^2m^{\cancel{3}}}{2r^2\cancel{m^2}v^2}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$\downarrow$$

$$E = -\frac{1}{2}GMm/r$$

why 椭圆 ? → detailed calculation

$$E = \frac{1}{2} m V_r^2 + \frac{1}{2} \frac{L^2}{m r^2} - \frac{GMm}{r} \Rightarrow$$

$$V_r^2 + \frac{L^2}{m^2} \frac{1}{r^2} - \frac{2GM}{r} - \frac{2E}{m} = 0$$

$$f\left(V_r, \frac{1}{r}\right) = 0$$

(refresh: what is V_r, r)

$$V_r^2 + a^2 \left(\frac{1}{r} - \beta\right)^2 = \Omega^2$$

$$a = \frac{L}{m}, \quad \beta = GM/a^2 = \frac{GMm^2}{L^2}$$

$$\Omega^2 = \frac{G^2M^2m^2}{L^2} + \frac{2E}{m}$$

平方配比

$$\begin{cases} V_r = \Omega \sin\varphi \\ \frac{1}{r} - \beta = \frac{\Omega}{a} \cos\varphi \end{cases}$$

求解 ansatz

(※2)

差角度 π (无妨)

比较 Berkeley P301 第一个等式：$\cos\varphi$ 前差一个负号

$$\frac{1}{r} = \frac{GM^2m_2}{J^2}\left[1 - \left(1 + \frac{2EJ^2}{G^2M^3m_2^2}\right)^{\frac{1}{2}} \cos\theta\right]$$

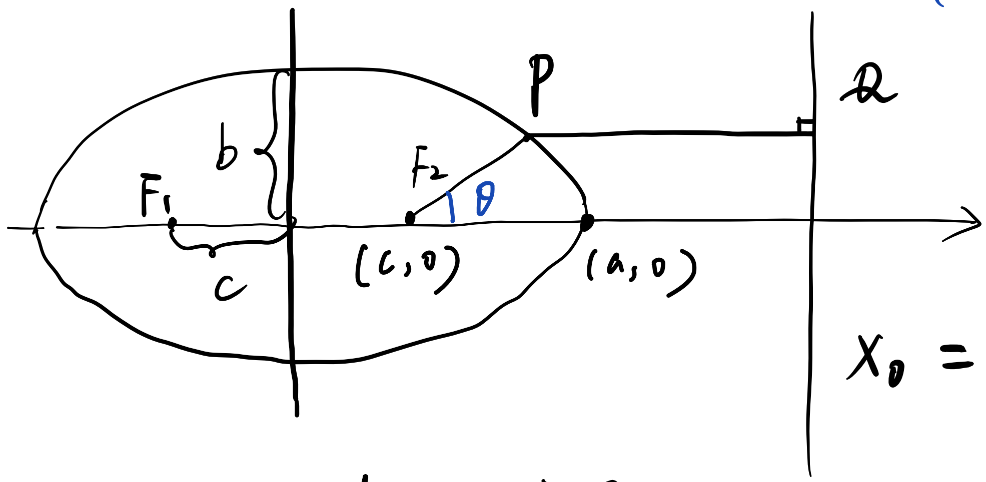
$L \to J$, $M \to m_2$, $m \to M$, $\varphi \to \pi + \theta$

review: ellipse

$0 < e = \frac{c}{a} < 1$

(eccentricity)

离心率



$X_0 = \frac{a^2}{c}$

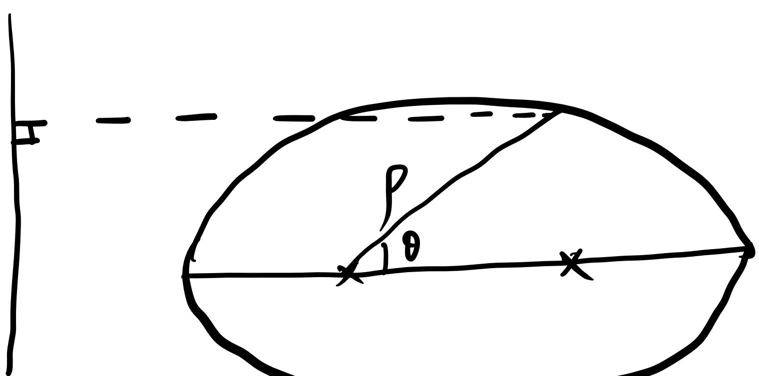
polar coordinate view: F_2 为原点

$$\frac{F_2P}{PQ} = \frac{\rho}{\frac{a^2}{c} - \rho\cos\theta - c} = \frac{c}{a} = e$$

$$\rho = a(1-e^2) - \rho e\cos\theta \qquad (a^2 = b^2 + c^2)$$

$$\rho = \frac{a(1-e^2)}{1 + e\cos\theta}$$

如果以左焦点为参考 $\rightarrow \rho = \frac{a(1-e^2)}{1 - e\cos\theta}$



$$\frac{\Omega}{\gamma\beta} = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$$

$$= e = \frac{c}{a}$$

(eccentricity)

离心率

Trajectory : $r(\theta) = ?$ back to (*1)

$$\omega = \frac{L}{mr^2}$$

$\frac{1}{2} m v_\theta^2 = \frac{1}{2} m \omega^2 r^2 = \frac{L^2}{2mr^2}$

$$\frac{d\theta}{dt} = \frac{L}{mr^2}$$

或参照 Berkeley P299
[注：此页为 ZXF 老师的推导，更简洁一些]

$$\frac{d\theta}{dt} dr = \frac{L}{mr^2} dr$$

$$v_r \, d\theta = -\frac{L}{m} d\left(\frac{1}{r}\right)$$

Ω, ∂ 是常数
E 守恒, L 守恒
(有心力场)

$$= -\frac{L}{m} d\left(\frac{1}{r} - \beta\right)$$

$$\Omega \sin\varphi \, d\theta = -\frac{L}{m} d\left(\frac{\Omega}{\partial}\cos\varphi\right)$$

$$= -\frac{L}{m}\frac{\Omega}{\partial}(-\sin\varphi) \, d\varphi$$

$$= \Omega \sin\varphi \, d\varphi \quad \left(\partial = \frac{L}{m}\right)$$

$$d\theta = d\varphi$$

$$\theta = \varphi + \varphi_c$$ (由初始条件确定)

$$\frac{1}{r} = \beta + \frac{\Omega}{\partial}\cos(\theta - \varphi_c)$$

$$= \beta(1 + e\cos(\theta - \varphi_c))$$

已知 $r(\theta)$ $\downarrow$

量纲

$$a = \frac{GMm}{-2E} \quad [M]$$

$$\beta = \frac{GMm^2}{L^2} \quad [M^{-1}]$$

here,

$$\beta = \frac{1}{a(1-e^2)}$$

椭圆轨道方程

$$\omega = \frac{d\theta}{dt} = \frac{L}{mr^2}$$

$$dt = \frac{mr^2}{L} d\theta$$

$$T = \int_0^{2\pi} \frac{m r(\theta)^2}{L} d\theta$$

$$= \int_0^{2\pi} \frac{m}{L} \frac{1}{\left[\beta + \frac{\Omega}{2}\cos(\theta - \varphi_c)\right]^2} d\theta$$

查积分表

换一种思路

$$\frac{ds}{dt} = \frac{L}{2m}$$ （掠面速度）

$$\int ds = \int \frac{L}{2m} dt$$

$$\pi ab = \frac{L}{2m} T$$

$$b = \sqrt{\frac{L^2}{-2Em}}$$

$$a = \frac{GMm}{-2E}$$

$$T = \frac{2m\pi ab}{L}$$

$$\frac{T^2}{a^3} = \left(\frac{2m\pi}{L}\right)^2 a^2 b^2 / a^3$$

$$= \left(\frac{2m\pi}{L}\right)^2 b^2/a$$

$$= \left(\frac{2m\pi}{L}\right)^2 \frac{L^2}{\cancel{-2E}\, m} \frac{\cancel{-2E}}{GMm}$$

$$= \frac{4\pi^2}{GM} \quad (\text{与 } m \text{ 无关})$$

云垂海立

此中有真意，欲辨已忘言。

$$\frac{1}{r} = \frac{1}{se}(1 - e\cos\theta)$$

S: the scale of the figure

Hyperbola	$e > 1$
parabola	$e = 1$
Ellipse	$0 < e < 1$
Circle	$e = 0$

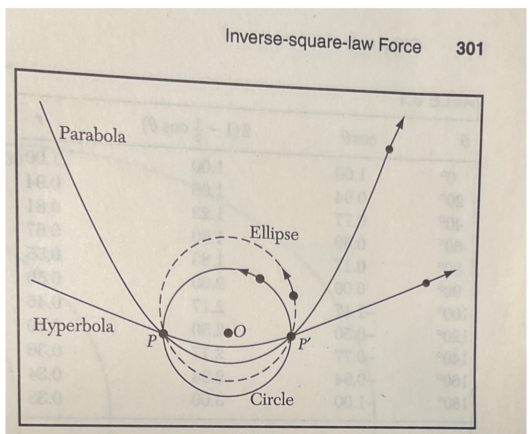


FIG. 9.13 Orbits of particles with the same mass M and angular momentum $\mathbf{J}$, but different energies E, about fixed center of force O. The orbits all cross at P and P'

Orbit	Eccentricity
Circle	$e = 0$ } $E < 0$
Ellipse	$e = \frac{1}{3}$ } $E < 0$
Parabola	$e = 1, E = 0$
Hyperbola	$e = 3, E > 0$

双曲线

some points:

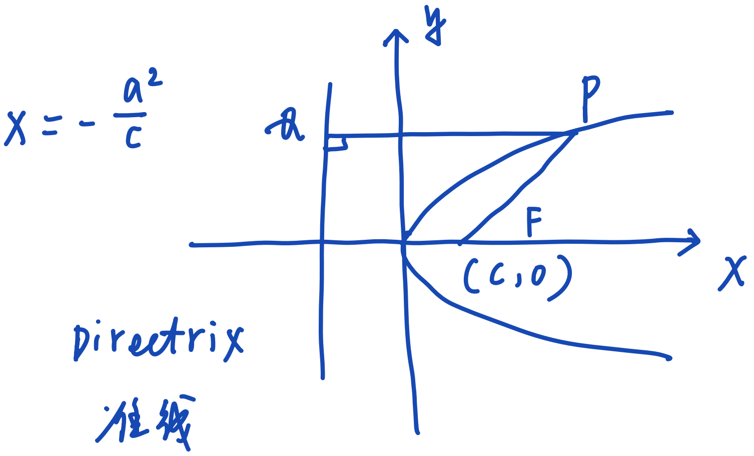
1 $V_r = 0$，只有一个交点，轨道与距离固定

圆周运动。

2 $se = a(1-e^2)$ （左焦点）

$$s = \frac{a}{e} - ae$$

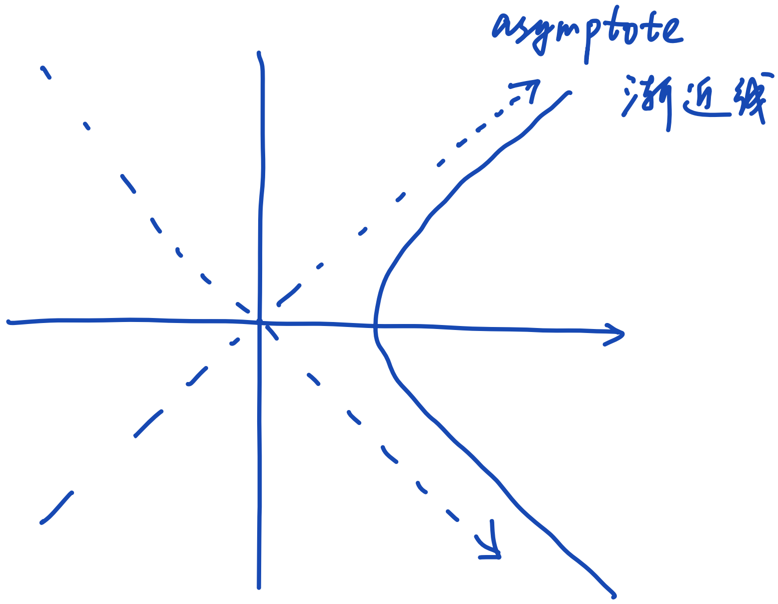
3 $e = 1$，抛物线 $(a = c)$



$y^2 = 4cx$

$|FP| / |PQ| = e$

4 $e > 1$ 双曲线



5. Conic Section :

Conic section or sections of a cone are curves obtained by the intersection of a plane and cone.