如何分析变量关系 (Newton / Leibnitz)

two sets : A and B

maps : F : $A \to B$

$a \mapsto F(a)$

$F(a) \equiv b \in B$

F : a rule, assigning to each element a of A a (-4) corresponding element b of B

A : domain (域)

B : codomain (陪域)

$a \in A$ ↓ argument

$F(a)$ ↓ image (element)

Differentiation of one-dimensional functions

Definition

Let $f: R \to R$

$y \mapsto f(y)$ be a smooth function.

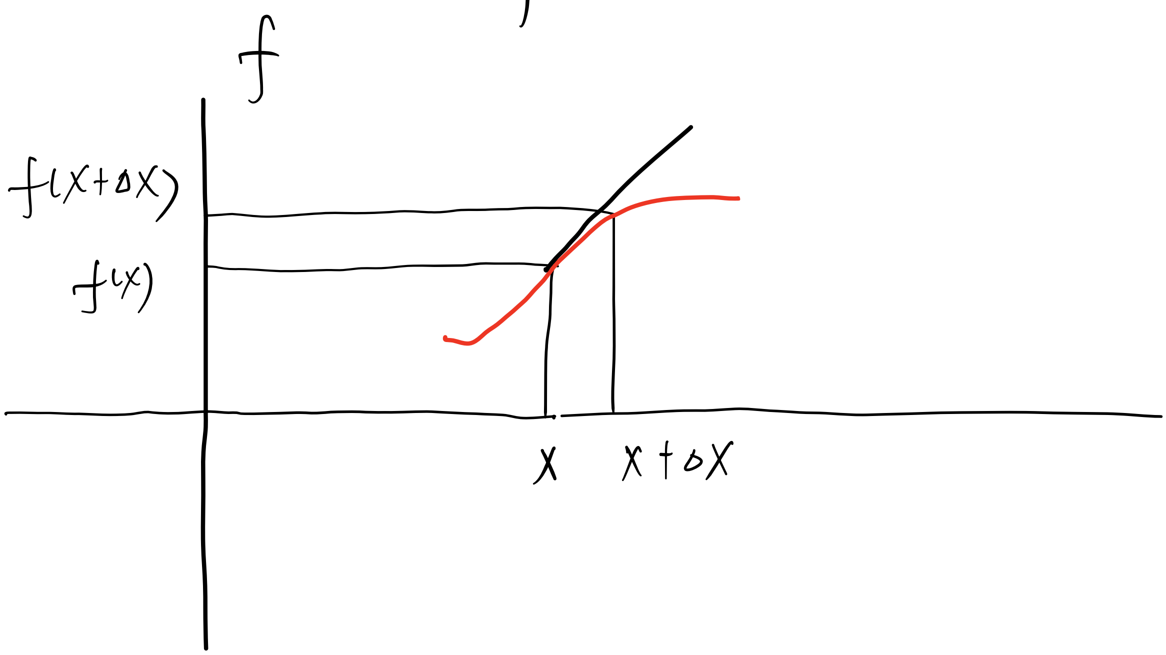
$$f'(x) = \frac{df(x)}{dx} \equiv \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(f(x+\Delta x) - f(x)\right) \quad (I)$$

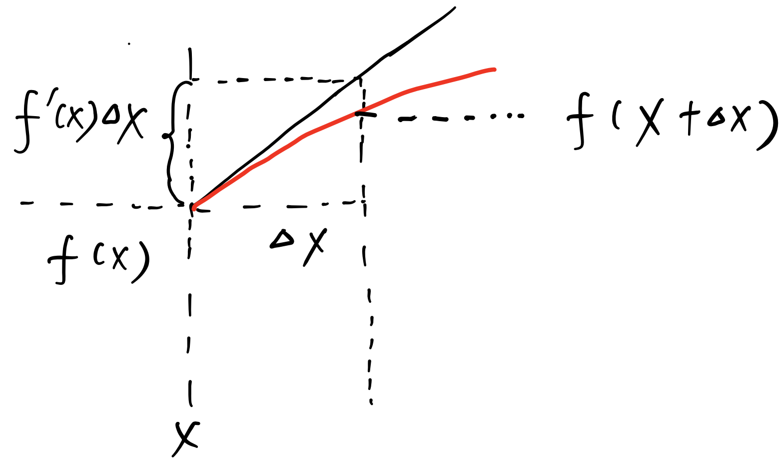
derivative of $f(x)$ with respect to x

if $f'(x)$ determines the slope of f at x

just ok!

A more powerful interpretation goes as follows:





Δx : small but not infinitesimally small

$$\frac{1}{\Delta x}\left(f(x+\Delta x) - f(x) \right) \simeq f'(x)$$

only in the limit $\Delta x \to 0$

the approximate equality becomes exact

$$f(x+\Delta x) \simeq f(x) + f'(x)\,\Delta x$$

setting $x + \Delta x = y$, i.e.

$$f(y) \simeq f(x) + f'(x)(y-x)$$

y is in the immediate neighborhood of x

All derivatives represent local approximations of functions by linear functions

keep this point in mind, no difficulty in understanding even very fancy derivative operations

E.g. $\sin\phi \to \sin(\phi+\delta)$

$$= \sin\phi \underbrace{\cos\delta}_{=1} + \underbrace{\sin\delta}_{=\delta} \cos\phi \qquad \delta \to 0$$

$$= \sin\phi + \delta\cos\phi$$

$$(\sin\phi)' = \frac{d\sin\phi}{d\phi} = \lim_{\delta\to 0}\frac{\sin(\phi+\delta)-\sin\phi}{\delta}$$

(here, $\delta \equiv \Delta\phi$)

$$= \cos\phi$$

课堂练习：$(\cos\phi)' = ?$

$$\left(\frac{1}{x}\right)' = ?$$

$$\Delta f(x) = \frac{1}{x+\Delta x} - \frac{1}{x} = \frac{-\Delta x}{x(x+\Delta x)}$$

$$\lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{-\Delta x}{x^2\left(1+\frac{\Delta x}{x}\right)\Delta x}$$

$$= -\frac{1}{x^2}$$

generalization

$\downarrow$ derivative of higher order

$$\frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{df(x)}{dx}\right)$$

$$\frac{d^n f(x)}{dx^n} = \frac{d^n}{dx^n} f(x) =$$

$$\underbrace{\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx} \cdots \left(\frac{d}{dx} f(x)\right)\right)\right)}_{n \text{ factors}}$$

(The notation)

$$f^n(x) \equiv \frac{d^n}{dx^n} f(x) \quad (\text{量纲同 } [x^n])$$

△ sum rule

$$(f(x)+g(x))' = f'(x)+g'(x)$$

△ product rule

$f \quad g$

$$\frac{d(fg)}{dx} = \frac{df(x)}{dx} g(x) + f(x)\frac{dg(x)}{dx}$$

product rule proof:

$$f(x+\Delta x)\, g(x+\Delta x) - f(x)\, g(x)$$

$$= [f(x+\Delta x) - f(x)]\, g(x+\Delta x)$$

$$+ f(x)\, [g(x+\Delta x) - g(x)]$$

then
$$\frac{d(fg)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} \lim_{\Delta x \to 0} g(x+\Delta x)$$

$$+ f(x) \lim_{\Delta x \to 0} \frac{\Delta g(x)}{\Delta x}$$

$$= f'(x)\, g(x) + f(x)\, g'(x)$$

前提：各自的极限存在

比如 $\cos(\frac{1}{x})$ 极限不存在

或

$\Delta(fg) = f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)$

$= (f(x)+\Delta f(x))\,(g(x)+\Delta g(x))$

$- f(x)\,g(x)$

$= \Delta f(x)\,g(x) + f(x)\,\Delta g(x) + \Delta g(x)\,\Delta f(x)$

$$\lim_{\Delta x\to 0}\frac{\Delta(fg)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta f(x)}{\Delta x}g(x)$$

$$+ \lim_{\Delta x\to 0} f(x)\frac{\Delta g(x)}{\Delta x}$$

$$= f'(x)\,g(x) + f(x)\,g'(x)$$

△ chain rule

$$\frac{d(f(g(x)))}{dx} = \frac{df(y)}{dy}\Big|_{y=g(x)} \frac{dg(x)}{dx}$$

composition: 复合

Given two maps, $F: A \to B$ and $G: B \to C$, the composition of the two is defined by

$$G \circ F : A \to C$$

(PS L3)

Alexand Altland

& Jan von Delft

$$a \mapsto G(F(a))$$

[Two examples:

① $$\frac{d f(ax)}{dx} = a \frac{df(y)}{dy}\Big|_{y=ax}$$

② $$\frac{d}{dx} \frac{1}{g(x)} = -\frac{1}{g(x)^2} \frac{dg(x)}{dx}$$

choice: $f(y) = \frac{1}{y}$ $\quad y = g(x)$

如何理解极限：（高阶内容）

1° 一尺之棰，日取其半，万世不竭

《庄子·天下篇》

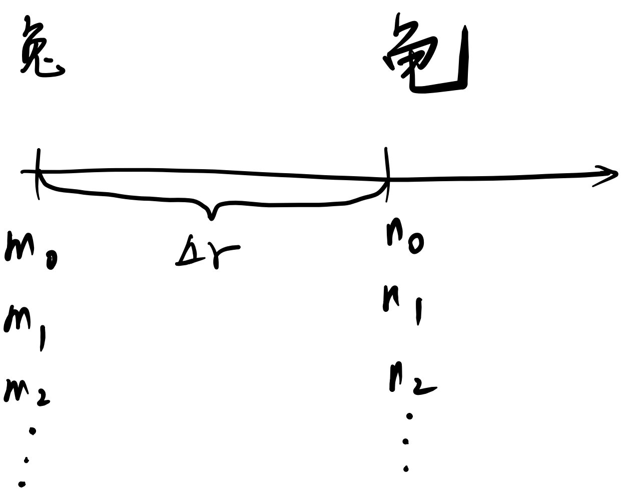
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\cdots$$

$$=1$$ （取完了吗？）

$$e=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

2° 龟兔赛跑的悖论

位置 $\{m\}\{n\}$ 在时间上的序列



$n_0 - m_0 = \Delta r$

$\Delta t_1 = \dfrac{\Delta r}{v_{兔}}$

$\Delta t_2 = \dfrac{\Delta r_1}{v_{兔}}$

$\Delta t_3 = \dfrac{\Delta r_2}{v_{兔}}$

兔子总需要花时间去到达乌龟上一个时间点的位置，永远追不上？

$\Delta r_1 = v_{龟}\Delta t_1$

$\Delta r_2 = v_{龟}\Delta t_2$

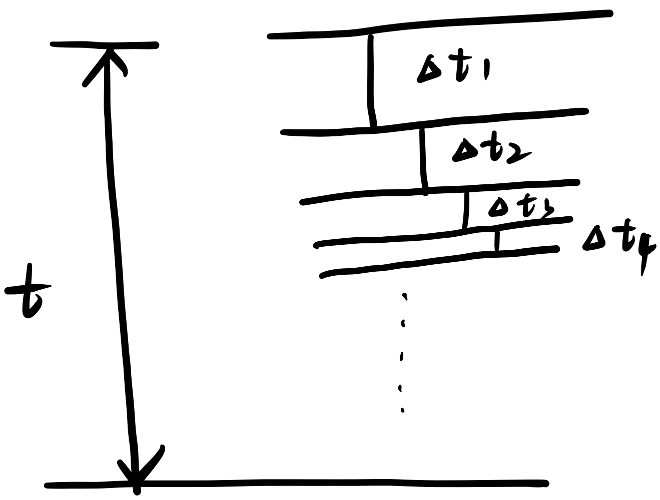
Zeno's paradox

(芝诺时间)

无限步骤 $\neq$ 无限时间

Δt_n $\quad n \to \infty$ 无限步骤

$$\sum_{n=1}^{\infty} \Delta t_n = t = \frac{\Delta r}{V_{\text{兔}} - V_{\text{龟}}}$$



无穷数列 $\{\Delta t_n\}$

求和极限

是有限值 t

Derivative of inverse functions (拓展)

$$y = f(x)$$

$$x = f^{-1}(y)$$

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}}$$

$$\frac{df^{-1}(y)}{dy} = \frac{1}{\left.\frac{df(x)}{dx}\right|_{x=f^{-1}(y)}}$$

$$\frac{d}{dy}(\arcsin y) = \frac{1}{\sin' x\,\big|_{x=\arcsin y}}$$

$$= \frac{1}{\cos(\arcsin y)}$$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin y)}}$$

$$= \frac{1}{\sqrt{1-y^2}}$$

$y = \sin x$

$x = \arcsin y$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

$$= \frac{1}{\cos x} \rightarrow y \text{ 是自变量}$$

$$= \frac{1}{\cos(\arcsin y)}$$

$$= 1/\sqrt{1-y^2}$$

Newton's method (牛顿方法)

linear approximation (Find $f(x)$)

At $x = a$ $\quad \frac{df}{dx} = f'(a) \approx \frac{f(x)-f(a)}{x-a}$

$$f(x) \approx f(a) + (x-a)f'(a)$$

Slove $f(x) = 0$

$$x - a \approx \frac{f(x)-f(a)}{f'(a)} = -\frac{f(a)}{f'(a)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{(iterative)}$$

eg.

Find $\sqrt{9.06} = ?$

set $x = \sqrt{9.06} \Rightarrow x^2 = 9.06 \Rightarrow$

$$x^2 - 9.06 = 0$$

solve $f(x) = x^2 - 9.06 = 0$

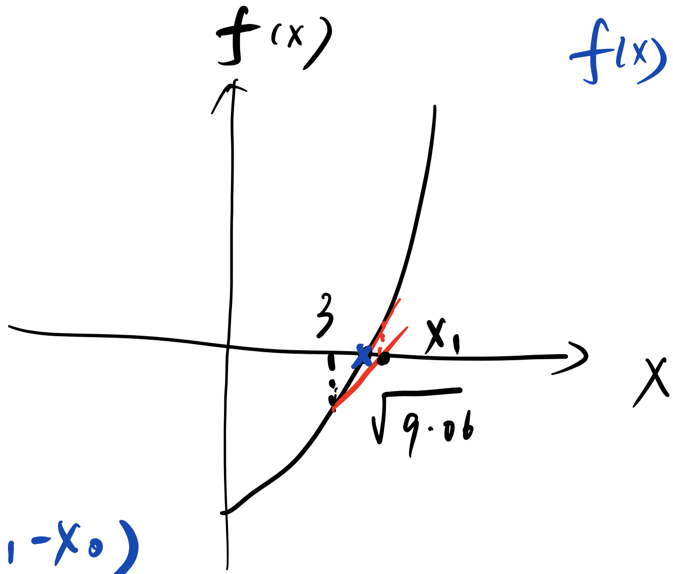
$$x = ?$$

$$x^* = 3.00998338658482\ldots$$

Let us start from $x_0 = 3$

$1°$ 初始估算点 $x_0 = 3$, $f(x_0) = -0.06$

$f'(x) = 2x \qquad f'(x_0) = 2 \times 3 = 6$



$f(x) = x^2 - 9.06$

(x_1 : 线性近似与 x 轴交点)

$\therefore f(x_1) = 0$

$f(x_1) - f(x_0)$

$\approx f'(x_0)(x_1 - x_0)$

$x_0 \to x_1$

$$x_1 - x_0 = \frac{-f(x_0)}{f'(x_0)} = \frac{0.06}{6}$$

$x_1 = 3.01$ (更好的估算点) 相较于 x_0

$(3.01)^2 = 9.0601$

$2°$ $x_1 = 3.01$

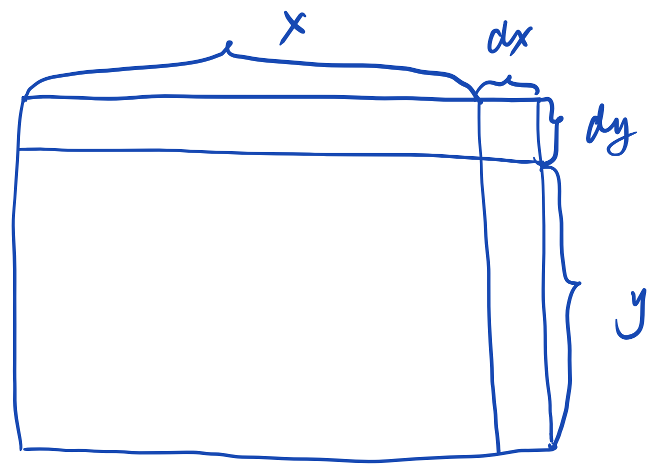
$$x_2 - x_1 = \frac{-f(3.01)}{2 \times 3.01} = \frac{-0.0001}{6.02}$$

$x_1 \to x_2$

$x_2 = 3.00998$

$\cdots \longrightarrow x_n$

两个变量 （有点难）



$$D(xy) = (x+dx)(y+dy) - xy = x\,dy + y\,dx + dx\,dy$$

$(O^2(\varepsilon))$

$$f(x, y) = xy \text{ (面积)}$$

$$df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

partial derivative：偏导

[后记] 从数集出发，一元函数是从数集到数集的映射，二元函数是从两个数集的笛卡尔集到数集的映射

微分 dx 是个函数

导数 $f'(x_0) = \lim\limits_{\delta \to 0} \dfrac{f(x_0+\delta)-f(x_0)}{\delta}$

是一个数

如果 $f'(x_0)$ 存在，for $\forall x_0 \in \{x\}$

$f'(x)$ 是导函数：关于自变量 x 的一元函数

如果 $f(x)$ 可导，那么

$y = f(x)$ 的微分 $dy = df(x) = f'(x)dx$

这里的 dy 是二元函数，一个自变量是 x，一个自变量是 dx（dx 仅仅记号特别）

可视为 $df(x) = g(x, w) = f'(x)\,w$

$(w \equiv dx)$

所谓微分 dy 是关于 dx 的线性函数。

考虑 x， dy 是自变量 x 和自变量 dx 的二元函数

d ： 微分算子

$$f(x) \xrightarrow{d} df(x) = f'(x)\,dx$$

可导一元函数 二元函数

术语：

微分 differential adj & n

差分 difference

积分 integral

导数 derivative

~~y 对 x 的导数~~

$$\underbrace{dy}_{y\text{的微分}} = y'(x)\,\underbrace{dx}_{x\text{的微分}}$$

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