oscillation

① $m\ddot{x} + kx = 0 \Rightarrow$

$$\ddot{x} + w_0^2 x = 0$$

② $\ddot{x} + w_0^2 x = F \cos wt$

②' $\ddot{x} + w_0^2 x + \gamma \dot{x} = 0$

③ $\ddot{x} + w_0^2 x + \gamma \dot{x} = F \cos wt$

$\downarrow$ in general

$$\ddot{x} + w_0^2 x + \gamma \dot{x} = F(t)$$

$$F(t) = \sum_{n=-\infty}^{\infty} F_n e^{i w_n t}$$

$$w_n = \frac{2\pi n}{T} \quad (n \in \mathbb{Z})$$

Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\pi} = -1 \quad , \quad e^{i2\pi} = 1$$

e^{ix} 比 $\cos x$ 求导运算方便

$$\frac{d^2x}{dt^2} + \omega_0^2 x = F\cos\omega t$$

怎么解?

$\uparrow$

(m 已吸收进 F)

$$\frac{d^2x_r}{dt^2} + \omega_0^2 x_r = F\cos\omega t \quad ①$$

$$\frac{d^2x_i}{dt^2} + \omega_0^2 x_i = F\sin\omega t \quad ②$$

$$x = x_r + i x_i$$

① + ② × i $\Rightarrow$

$$\frac{d^2x}{dt^2} + \omega_0^2 x = F e^{i\omega t}$$

分析： 只能含有 x 的 0 或 1 次幂，

否则实虚部将混杂。

$$\frac{d(e^{i\omega t})}{dt} = i\omega\, e^{i\omega t}$$

$$\frac{d^2(e^{i\omega t})}{dt^2} = (i\omega)^2 e^{i\omega t}$$

令 $x = x_0 e^{i\omega t}$

$$x_0 (i\omega)^2 \cancel{e^{i\omega t}} + \omega_0^2 x_0 \cancel{e^{i\omega t}} = F \cancel{e^{i\omega t}}$$

$$x_0 = \frac{F}{\omega_0^2 - \omega^2}$$

$\omega_0 < \omega$ ， 相差为 π

$\omega_0 > \omega$ ， 相差为 0

$$|x_0| = C \quad |F|$$

C: factor

$\omega \sim \omega_0$，极强响应

理想驱动 → 现实阻尼

$$\frac{d^2x}{dt^2} + \boxed{\gamma \frac{dx}{dt}} + \omega_0^2 x = F\cos\omega t$$

friction

$$[(i\omega)^2 x_0 + \gamma(i\omega)x_0 + \omega_0^2 x_0]e^{i\omega t} = Fe^{i\omega t}$$

$$x_0 = \frac{F}{\omega_0^2 - \omega^2 + i\gamma\omega}$$

令 $R = \frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega}$

$$x_0 = RF$$

F是实数，这说明 x_0 与 F 有相位之差

$$R = \rho e^{i\theta}$$

$$\frac{1}{R} = \frac{1}{\rho} e^{-i\theta}$$

ρ, θ ： 响应的大小和相移

★ 插入基本知识 ： 没移动

$$(x+iy) R$$

$$re^{i\phi} \rho e^{i\theta}$$

模长 ： $r \to \rho r$

辐角 ： $\phi \to \phi + \theta$

here,

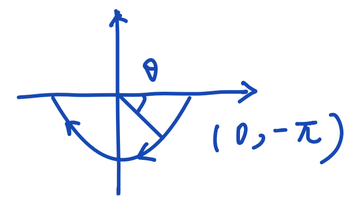
$$\rho^2 = \frac{1}{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2} \quad \boxed{(*)}$$

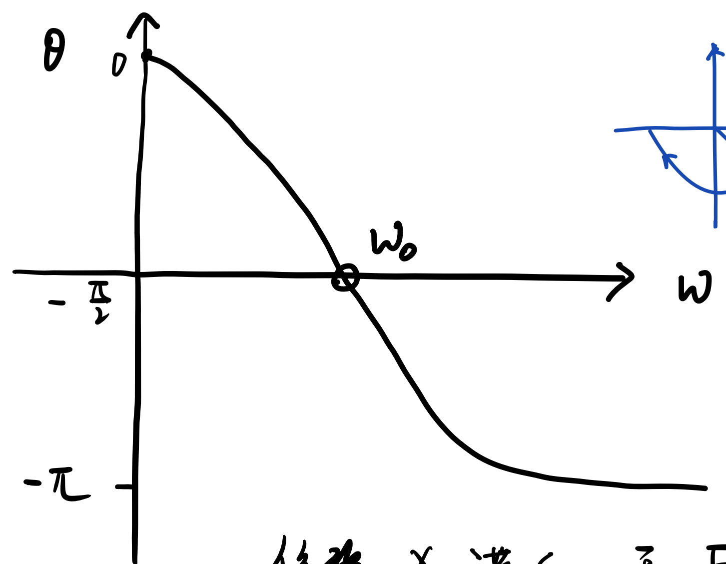
$$\tan\theta = -\gamma\omega / \omega_0^2-\omega^2$$

$\mathrm{Im}(R) \sim -\gamma\omega < 0$ （三，四象限）

$\mathrm{Re}(R) \sim \omega_0^2 - \omega^2$

θ为负数





位移 X 滞后于 F

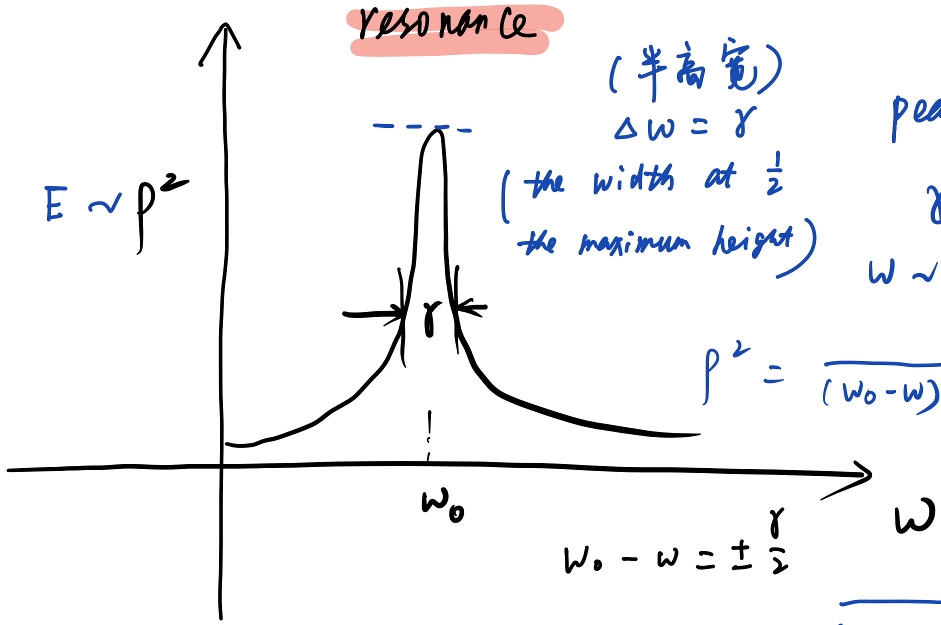
resonance

(半高宽)

$\Delta\omega = \gamma$

(the width at $\frac{1}{2}$ the maximum height)

$E \sim \rho^2$



$\omega_0 - \omega = \pm\frac{\gamma}{2}$

peak location:

$\gamma \to 0$

$\omega \sim \omega_0 \to$ infinity

$$\rho^2 = \frac{1}{(\omega_0-\omega)^2(\omega_0+\omega)^2 + \gamma^2\omega_0^2}$$

$$\Downarrow$$

$$\frac{1}{(\omega_0-\omega)^2 \, 4\omega_0^2 + \gamma^2\omega_0^2}$$

$$= \frac{1}{4\omega_0^2} \cdot \frac{1}{[(\omega_0-\omega)^2 + \gamma^2/4]}$$

$$R = \omega_0^2 - \omega^2 + i\gamma\omega$$

$$\approx 2\omega_0(\omega_0 - \omega + i\gamma/2)$$

$(\text{if } \gamma \ll \omega_0, \ \omega \approx \omega_0)$

$Q = \omega_0/\gamma$ (品质因子)

γ 很小， $\omega = \omega_0$ 附近，"共振峰"

如果 go to details.

where is the maximum ? $\omega = \omega_0$?

believe in math !

take the derivative with respect to ω for eqn. $\boxed{(*)}$

分母求导： $2(\omega^2 - \omega_0^2)\, 2\omega + \gamma^2\, 2\omega = 0$

$$\omega^2 = \omega_0^2 - \frac{1}{2}\gamma^2$$

代回 ρ^2, 分母 $= \left(\frac{1}{2}\gamma^2\right)^2 + \gamma^2\left(\omega_0^2 - \frac{1}{2}\gamma^2\right)$

$$= \omega_0^2\gamma^2 - \frac{1}{4}\gamma^4$$

回到问题本身 $\theta \in (-\pi, 0]$

$$x = x_0 e^{i\omega t}$$

$$x = \mathrm{Re}\left(\rho F e^{i\omega t} e^{-i|\theta|}\right)$$

"0" for "$\omega = 0$"

$$= \rho F \cos(\omega t - |\theta|)$$

Note: 到目前为止，我们根本没使用初始条件，何处安放？

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

可以把满足这个方程的 x 补进去

在 $D \equiv \frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2$

下 仅仅是补"0"而已

求解一下这个有阻尼无驱动的情形：

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0$$

$$x = x_0 e^{i\omega t}$$

$$(-\omega^2 + i\omega\gamma + \omega_0^2)\, x_0 e^{i\omega t} = 0$$

求解一元二次方程

$$\omega^2 - i\omega\gamma - \omega_0^2 = 0$$

$$\omega = \frac{i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2}$$

$$\downarrow \quad \gamma = 2\beta$$

$$\omega = i\beta \pm \sqrt{\omega_0^2 - \beta^2}$$

$\omega_0 > \beta$ under-damping

$$\omega_+ = \sqrt{\omega_0^2-\beta^2} + i\beta \quad (\equiv \omega_p)$$

$$\omega_- = -\sqrt{\omega_0^2-\beta^2} + i\beta$$

$$e^{i\omega t} = e^{-\beta t}\, e^{\pm i\omega_p t}$$

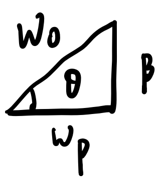
$$x = Ae^{i\omega_+ t} + Be^{i\omega_- t}$$

$x(0) = x_0, \quad v(0) = 0$ (initial condition)

$$x(0) = A + B = x_0$$

$$v(0) = i\omega_+ A e^{i\omega_+ t} + i\omega_- B e^{i\omega_- t} = 0$$

$$A(\omega_p + i\beta) + B(-\omega_p + i\beta) = 0$$



$$\omega_0 A e^{i\theta} + \omega_0 B e^{i(\pi-\theta)} = 0$$

$$Ae^{i\theta} = Be^{-i\theta} \Rightarrow |A| = |B|$$

联立 $\quad A + B = x_0 \Rightarrow A = B^*$

$$\Downarrow \quad Re(A) = Re(B) = \frac{x_0}{2}$$

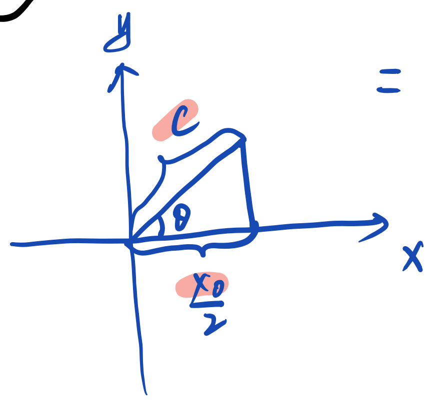
$$A = \frac{x_0}{1+e^{2i\theta}} = \frac{x_0 e^{-i\theta}}{2\cos\theta}$$

$$\downarrow \quad c = \frac{x_0}{2} \Big/ \cos\theta$$

则有 $x = e^{-\beta t}\left(c\, e^{-i\theta} e^{i\omega_p t} + c\, e^{+i\theta} e^{-i\omega_p t}\right)$

$$= c\, e^{-\beta t}\left[2\cos(\omega_p t - \theta)\right]$$

衰减型振荡



$$= \frac{x_0}{\cos\theta} e^{-\beta t} \boxed{\cos(\omega_p t - \theta)}$$

$\sin\theta = \beta / \omega_0$

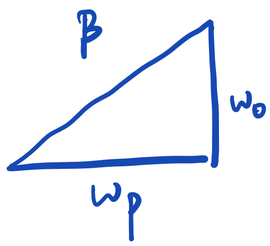
滞后效应

turn to $\boxed{\text{over - damping}}$

$\omega_0 < \beta$ $\qquad \omega_p = \sqrt{\beta^2 - \omega_0^2}$

$$\omega_\pm = i\beta \pm i\,\omega_p$$

$$x = A e^{i\omega_+ t} + B e^{i\omega_- t}$$



$$= A e^{-\beta t} e^{-\omega_p t} + B e^{-\beta t} e^{\omega_p t}$$

$$\beta > \omega_p$$

两种不同程度衰减的线性叠加

$$X_0 = X(t=0) = A + B$$

$$V_0 = \frac{dx}{dt}\Big|_{t=0} = i\omega_+ A + i\omega_- B = 0$$

$$A(\beta + \omega_p) + B(\beta - \omega_p) = 0$$

$$\beta(A+B) + \omega_p(A-B) = 0$$

$$\beta X_0 / \omega_p = B - A$$

$$A = \frac{X_0}{2}(1 - \beta/\omega_p), \quad B = \frac{X_0}{2}(1 + \beta/\omega_p)$$

$A < 0$, $B > 0$, 且有 $B > |A|$

$\longrightarrow$ critical damping

$\omega_p = 0$

$e^{-\beta t}$ is the only solution ?

Limit approaching !

it can be started from

either side of $\omega_p = 0$

Furthermore, $A e^{-\beta t}$

can not satisfy

$$\begin{cases} x(t=0) = x_0 \\ v(t=0) \neq 0 \end{cases}$$

$t e^{-\beta t}$ is another solution

analysis : two initial conditions

the initial position

the initial velocity

but, the acceleration can not be choosen, it is due to Newton's laws.

r_0, $\dot{r}_0$ $(= v_0)$ are fixed

the $\ddot{r}_0$ is fixed