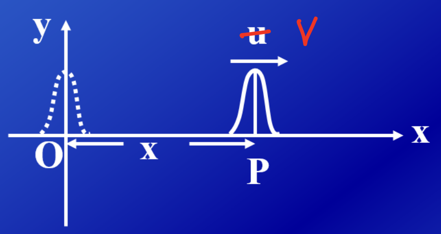
What is a traveling wave ?

Traveling wave



O: $y = f(t)$

P: $y = f\left(t - \frac{x}{\cancel{u}}\right)$ V

Wave propagates the "SHAPE" with speed ~~u~~ V

$\frac{\partial y}{\partial t}$: vibration $\frac{\partial y}{\partial x}$: shape

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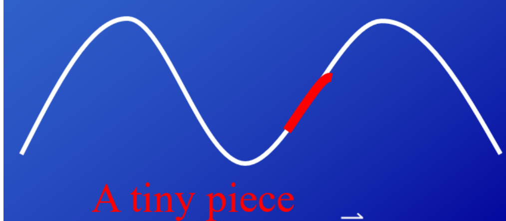
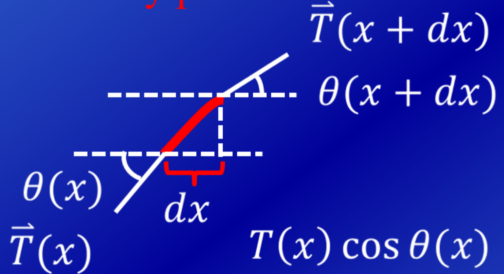
旅行的意义

Wave equation from equation of motion





Newton' 2nd law

X-direction:

$$T(x)\cos\theta(x) = T(x+dx)\cos\theta(x+dx)$$

Denote $T_x = T$ constant

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if $T_x(x+dx) \neq T_x(x)$

ρdx 质元将在水平方向加速。

Y-direction:

$$T_y(x+dx) - T_y(x) = (\rho dx)\frac{\partial^2 y}{\partial t^2}$$

$$T_y = T_x \tan\theta = T\frac{\partial y}{\partial x}$$

$$T\frac{\partial^2 y}{\partial x^2}dx = (\rho dx)\frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2} \Longrightarrow$$

$$\frac{T}{\rho} = v^2 \rightarrow v = \sqrt{\frac{T}{\rho}}$$

Wave equation

$$F_y = T_y(x+dx) - T_y(x)$$

$$z \equiv x \pm vt$$

$$f(z) = f(x \pm vt)$$

$$\frac{\partial f}{\partial x} = \frac{df}{dz}\frac{\partial z}{\partial x} = \frac{df}{dz}$$

$$\frac{\partial f}{\partial t} = \frac{df}{dz}\frac{\partial z}{\partial t} = \pm v \frac{df}{dz}$$

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{d}{dz}\frac{df}{dz} = \frac{d^2 f}{dz^2}$$

$$\frac{\partial}{\partial t}\frac{\partial f}{\partial t} = \pm v \frac{d}{dz}\left(\pm v \frac{df}{dz}\right) = v^2 \frac{d^2 f}{dz^2}$$

$$\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$$ (波动方程)

$\pm v$: travelling wave

旅行的速度

"旅行的意义"

the solution of wave function :

$$y = A\cos(\omega t - kx) = A\cos\omega\left(t - \frac{x}{v}\right)$$

$$\frac{\partial y}{\partial x} = Ak\sin(\omega t - kx)$$

$$\frac{\partial y}{\partial t} = -A\omega\sin(\omega t - kx)$$

$$\frac{\partial y}{\partial t} = -v\frac{\partial y}{\partial x}$$ （右行波）

$$T_y = -T\frac{\partial y}{\partial x} \sim -\frac{T}{\Delta x}\Delta y$$

$$= -k_s\Delta y$$ (k_s : spring coef)

与后面纵波的图像一样 （耦合谐振子）

k : wave number

$$k \equiv 2\pi/\lambda$$

$$\omega = 2\pi/T$$

波的传播速度: $v = \frac{\lambda}{T} = \omega/k$

$$(x, t) \longrightarrow (x + \Delta x, t + \Delta t)$$

$$\theta(x, t) = \theta(x + \Delta x, t + \Delta t)$$

不变量

经历 Δt 时间，相位信息传到了相隔 Δx 的地方。

$$\omega \Delta t = k \Delta x$$

$$\theta = \omega t - kx$$

$$d\theta = \frac{\partial \theta}{\partial t} dt - \frac{\partial \theta}{\partial x} dx$$

$$= \omega \, dt - k \, dx$$

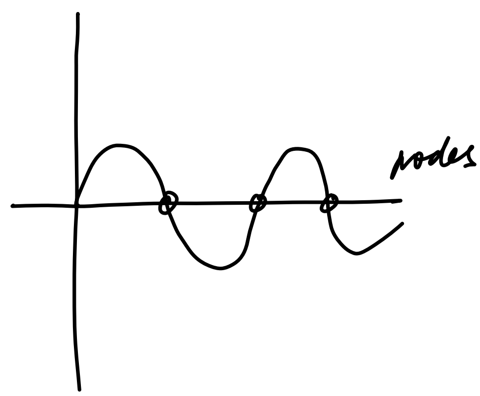
$$v_\theta = \left(\frac{dx}{dt}\right)_{[d\theta = 0]} = \frac{\omega}{k}$$

相位传播速度

Crest 波峰

Trough /trɔ:f/ 波谷

node 波节



关于 phase velocity

t_1 , x_1 两个时空点

t_2 , x_2

$\cos(\omega t - kx)$ $\theta = \omega t - kx$

θ 决定振动位置，相同 θ → 相同的 $A\cos\theta$

$\theta_1 = \theta_2$, i.e.,

$\omega t_1 - kx_1 = \omega t_2 - kx_2$

$d\theta = 0$

$dt = v\,dx$

$v_p = v$ 相位变化 (phase)

简谐波里，每个点的相位变化速度一样。

这一个点与整体讨论是一样的。

Another way :

σ : 应力 (以伸长为正方向) 

$$[-\sigma(x+\Delta x) + \sigma(x)]\,S = \rho S \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x} \cancel{\Delta x\, S} = \rho \cancel{S \Delta x} \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 y}{\partial t^2}$$

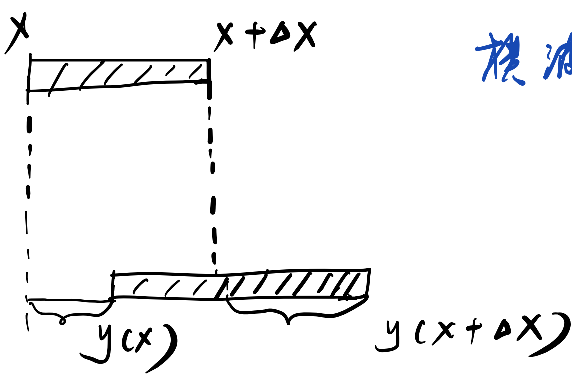
在此讨论

横波 纵波 (疏密波)

$y(x)$:

x处质元偏离

平衡位置位移



$$\Delta y = y(x+\Delta x) - y(x) \; : \; \text{deformation}$$

$$\boxed{\Delta y = \frac{\partial y}{\partial x} \Delta x}$$

(stress)

应力 杨氏模量

$$\frac{F}{S} = \underset{\text{应力}}{\sigma} = -\underset{\text{杨氏模量}}{Y} \frac{\Delta y}{\Delta x}$$

$$F = -\overset{K_s}{\frac{SY}{\Delta x}} \overset{\text{deformation}}{(\Delta y)}$$

$$E_p = \frac{1}{2}\left(\frac{SY}{\Delta x}\right)\Delta y^2 = \frac{1}{2}\left(\frac{SY}{\Delta x}\right)\left(\frac{\partial y}{\partial x}\right)^2 \Delta x^2$$

$$= \frac{1}{2} Y S \Delta x \left(\frac{\partial y}{\partial x}\right)^2$$

$$\sigma(x+\Delta x)S - \sigma(x)S = \rho S \Delta x \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial \sigma}{\partial x}\Delta x \cancel{S} = \rho \cancel{S} \Delta x \frac{\partial^2 y}{\partial t^2}$$

here

$$Y\frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}$$ (ρ 是体密度)

$$V = \sqrt{\frac{Y}{\rho}}$$

$$y = A\cos(\omega t - kx) = A\cos\omega\left(t - \frac{x}{v}\right)$$

$$E_p = \frac{1}{2} Y S \Delta x \left(\frac{\partial y}{\partial x}\right)^2 = \frac{1}{2} Y S \Delta x\, k^2 A^2 \sin^2(\omega t - kx)$$

$$E_k = \frac{1}{2} dm \left(\frac{\partial y}{\partial t}\right)^2 = \frac{1}{2} \rho \Delta x S \omega^2 A^2 \sin^2(\omega t - kx)$$

$$E_p = E_k \Longrightarrow E = E_k + E_p = \rho \Delta x S \omega^2 A^2 \sin^2(\omega t - kx)$$

(体积元)

能量密度 $$\frac{E}{V} = \rho \omega^2 A^2 \sin^2(\omega t - kx)$$

是时间的周期函数

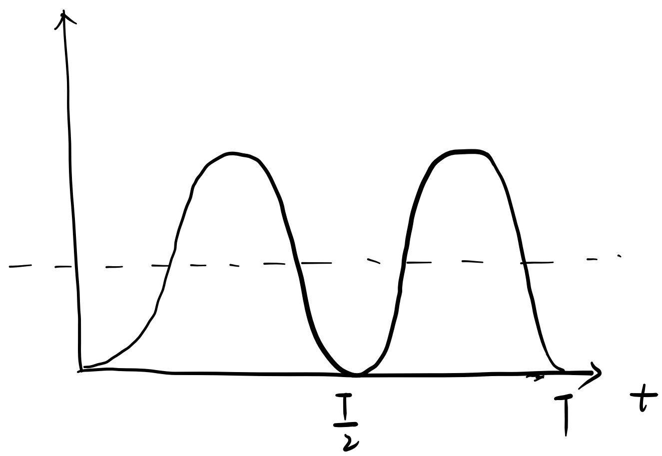
参照上面的推导：

每一个质元不是独立的弹簧，不会任何时刻独自能量守恒。

所有的弹簧之间彼此牵扯，各自在一个周期内经过人生的高低起落，平均能量相等

$$E \propto (1-\cos 2(\omega t - kx))$$

figure：



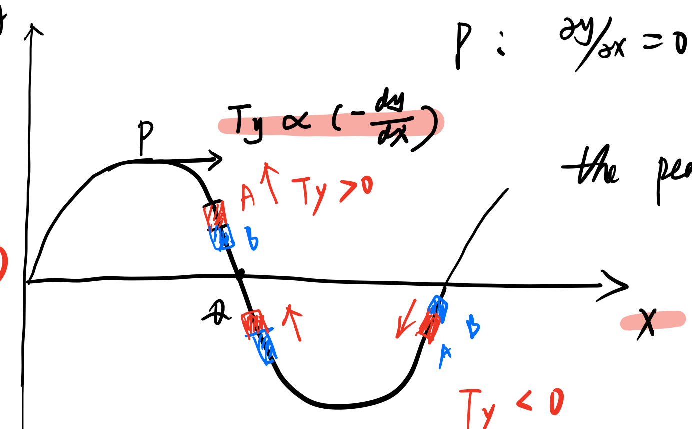
如果一个质元弹簧、时刻能量守恒，将不会有能量传播这样一件事情。右行波，左边质元一直对右边做正功。

Q：波与我们讨论过的简谐振子有何差别？

$$y = A\cos(\omega t - kx)$$

对于 Δx 这段弹簧而言

Fig (a)



P : $\partial y / \partial x = 0$, deformation $= 0$

$E_p = 0$

the peak of life, no velocity,

$E_k = 0$

$t_0 = \frac{T}{4} \Rightarrow y = A\cos(\frac{\pi}{2} - kx)$

$= A \sin kx$

(如左图)

always $\boxed{T_y \cdot \vec{v} > 0}$ (A对B做功)

P、a 曲线 $\frac{\partial y}{\partial x}$ 同号，

$$\boxed{\begin{array}{c}\text{在你之前的质元}\\ \text{永远对你作正功}\end{array}}$$

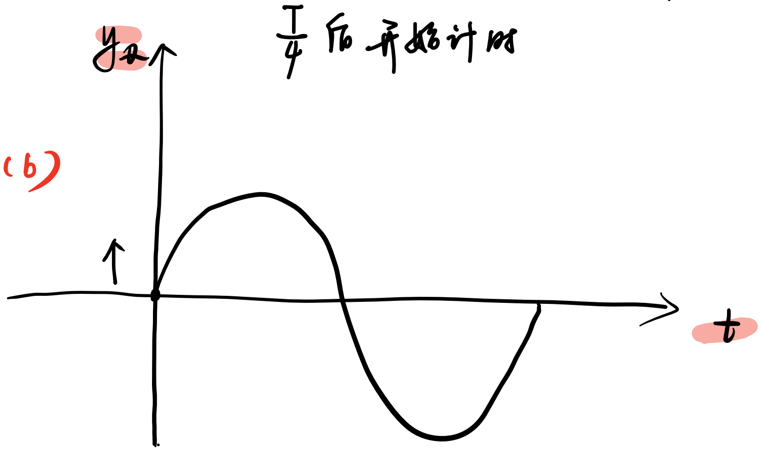
a 点 : $t_0 = \frac{T}{4}$, $x = \frac{\lambda}{2}$

$$y_a = A\cos(\frac{\pi}{2} - \pi + \omega t)$$

$$= A \sin \omega t$$

$\frac{T}{4}$ 后开始计时

Fig (b)



右行波

$$\frac{\partial y}{\partial t} = -v \frac{\partial y}{\partial x}$$

local dynamics

↓

shape

$\left.\frac{\partial y}{\partial t}\right|_{t=0}$ takes maximum

Q点：上图 $\left|\frac{\partial y}{\partial x}\right|$ 也最大， deformation 最大
Fig(a)
同时最大 (速度与形变)
而 P点是同时最小

review the energy of the mass element :

$$W = \rho S \Delta x \omega^2 A^2 \sin^2(\omega t - kx)$$

$$\frac{W}{S\Delta x} = w = \rho \omega^2 A^2 \sin^2(\omega t - kx)$$

$\downarrow$

energy density

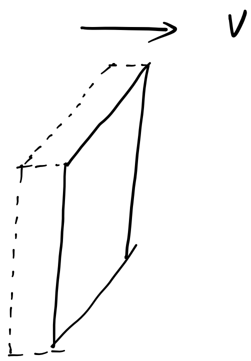
$$\langle w \rangle = \frac{1}{T}\int_0^T \rho \omega^2 A^2 \sin^2(\omega t - kx)\, dt$$

$$= \frac{1}{2}\rho \omega^2 A^2$$

travelling wave

signal transmit

$V = \frac{w}{k}$, $t = T$



$v\,dt\,s\,w$ 能量密度

dt 时间：穿过截面 s 的能量

比较电流强度

$I = nesv$

$I = \frac{q}{t} = \frac{nsle}{t} = nesv$

电荷流超率

$$p = \frac{v\,dt\,s\,w}{dt}$$ 瓦（功率）

$$= \boxed{ws}\,v$$

线能量密度　流超率.

$$\frac{\langle p\rangle}{S} = I = \langle w\rangle v$$ [瓦/米^2]

体能流密度　流超率

（波的强度）

一个周期内通过 S_1 和 S_2 的能量相等

$$I_1 S_1 T = I_2 S_2 T$$

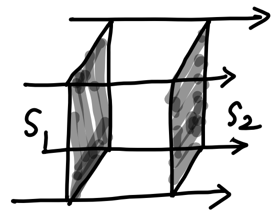
平面波 : plane wave (等相面是平面)

$$\frac{1}{2}\rho w^2 A_1^2 v s_1 T = \frac{1}{2}\rho w^2 A_2^2 v s_2 T$$

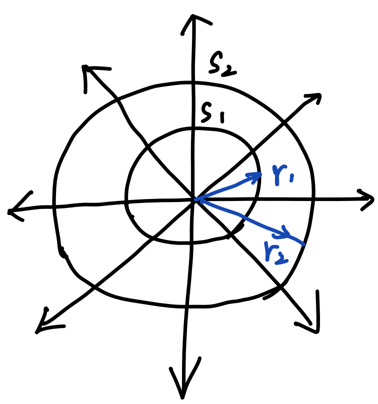
$S_1 = S_2$

$A_1 = A_2 = A$

(振幅)



球面波 : 等相面是球面



$$\frac{S_1}{r_1^2} = \frac{S_2}{r_2^2}$$

$$A_1^2 S_1 = A_2^2 S_2$$

$$\frac{S_1}{S_2} = \frac{r_1^2}{r_2^2} = \left(\frac{A_2}{A_1}\right)^2$$

$$A_2 r_2 = A_1 r_1$$

$$y = \frac{A}{r}\cos w\left(t - \frac{r}{v}\right)$$

一般情况下，会有波的吸收，转化为介质的内能或热。

standing wave :

$$y = A\cos(\omega t - kx) + A\cos(\omega t + kx)$$

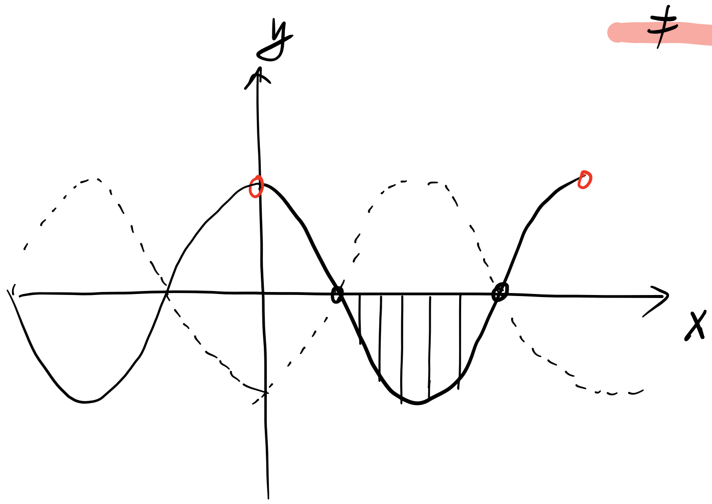
$$= 2A\cos\omega t\cos kx$$

$$= 2A\cos kx\cos\omega t$$

no travelling

$y(t+\Delta t,\ x+v\Delta t)$

$\neq y(t, x)$



amplitude : position dependent

birth rate

波节 : $E_k \equiv 0$

(注意区分角频率 ω)

$\langle W\rangle v - \langle W\rangle v = 0$

能量密度

$E_p + E_k = const$

在一个时间周期内

without energy transfer

standing, what about the tension?

go back to the superposition of the left / right travelling waves

$$\langle W \rangle_L = \langle W \rangle_R$$

$$W_L(t) \neq W_R(t)$$

— reply to 常天硕's 问

$$E_k = \frac{1}{2} \rho S \Delta x \left(\frac{\partial y}{\partial t}\right)^2 = 2 \rho S \Delta x A^2 w^2 \cos^2 kx \sin^2 wt$$

$$E_p = \frac{1}{2} YS/\Delta x \left(\frac{\partial y}{\partial x}\right)^2 \Delta x^2 = 2 Y S \Delta x A^2 k^2 \cos^2 kx \sin^2 wt$$

$$= 2 \rho S \Delta x A^2 w^2 \sin^2 kx \cos^2 wt$$

$$E = E_k + E_p = 2 \rho S \Delta x A^2 w^2 (\cos^2 kx \sin^2 wt + \sin^2 kx \cos^2 wt)$$

$$\langle E \rangle = \frac{1}{T} \int E \, dt = \rho S \Delta x A^2 w^2$$

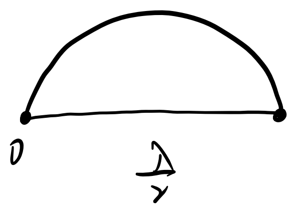
$$\langle E \rangle / S\Delta x = \rho A^2 w^2$$ （肯定对了，两倍于行波情形 两束行波对打）

平均能量密度

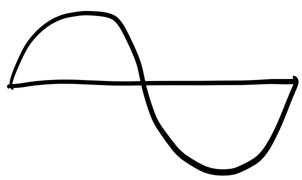
the superposition of two travelling waves

$\langle E_k \rangle = \langle E_p \rangle$

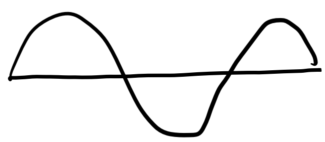
nodes :



$\frac{\lambda}{2} = L$



$\lambda = L$



$\frac{3}{2}\lambda = L$

$\sin kL = 0$

$kL = n\pi$

$k = n\frac{\pi}{L}$.

$$\lambda_n = \frac{2\pi}{k_n} = \frac{2L}{n}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{v k_n}{2\pi} \Leftarrow$$

$$= \frac{v}{2\pi} n$$ → integer

$$y(x,t) = \sum_n C_n \sin k_n x \cos \omega_n t$$

Fourier transformation → a special linear superposition

拍: $y = A\cos(w_1 t - k_1 x) + A\cos(w_2 t - k_2 x)$

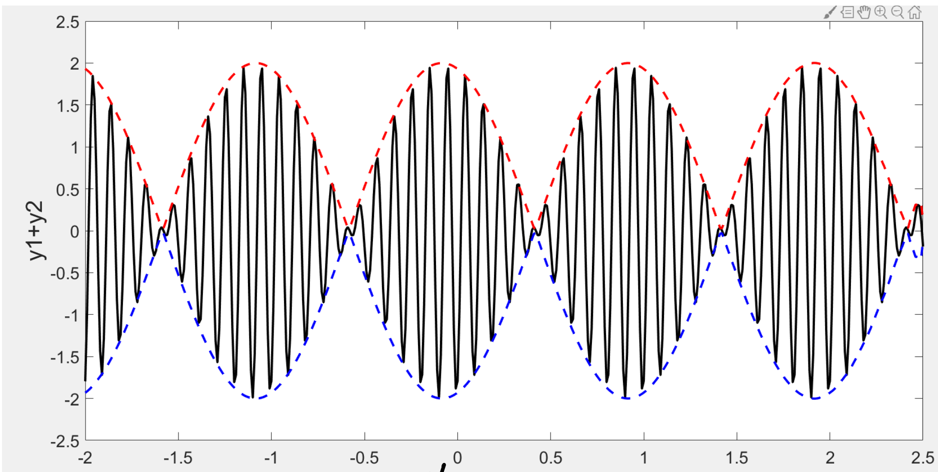
beat frequency

$$= 2A\cos\left(\frac{w_1+w_2}{2}t - \frac{k_1+k_2}{2}x\right)\cos\left(\frac{w_1-w_2}{2}t - \frac{k_1-k_2}{2}x\right)$$

$w_1 \approx w_2$

$$= A_g\cos\left(\frac{w_1+w_2}{2}t - \frac{k_1+k_2}{2}x\right)$$

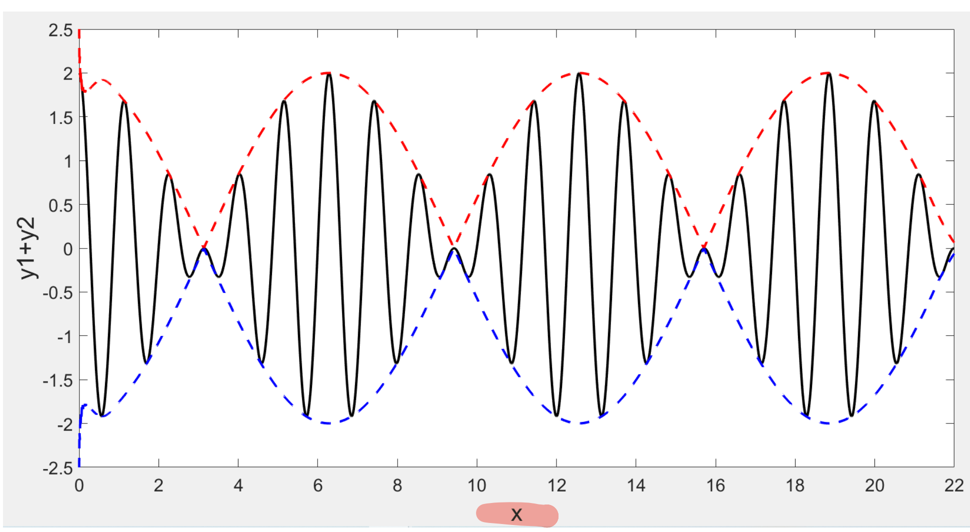
$x = 0.5$



E.g. $w_1 = 10 \times 2\pi$ $\quad k_1 = 6$ $\quad A = 1$

$w_2 = 11 \times 2\pi$ $\quad k_2 = 5$

$t = 1$



travelling wave : "envelope" velocity

A_g 缓变 $\omega_g = \left| \frac{\omega_1 - \omega_2}{2} \right|$

质元以 $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$ 快变

选型: group : 波群, 波包, 包络线 (envelope)

$$V_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega}{dk}$$

$$V_p = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\omega}{k}$$

Q: 听的是什么? 能听出 A_g, $-A_g$ 的差别吗?

听到的拍频 $f_{拍} = 2 \times \frac{|\omega_1 - \omega_2|}{2\pi}$

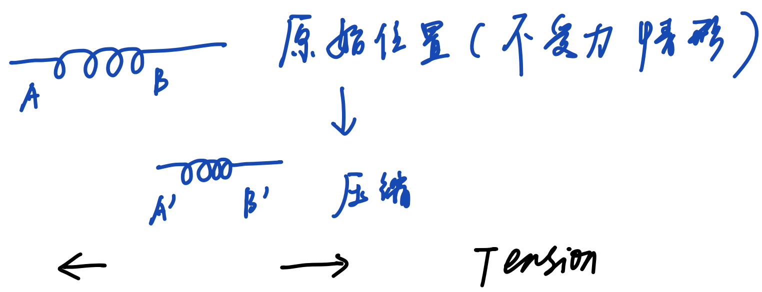
$= |\omega_1 - \omega_2| / \pi$

难点：

$$T \propto -\frac{\partial y}{\partial x}$$

用 $\frac{\partial y}{\partial x}$ 刻画，而非 y 本身

对一段弹簧而言，不能视为质点.



$x_{B'} > x_B$ ，对于右端点，

但弹簧所受的张力并非向左，而是向右。

不以各自端点偏离平衡位置来考虑受力，

而是以原长度为 Δx 的弹簧的形变量 Δy

的正负来考量 而 $\Delta y \neq x_{B'} - x_B$，$\Delta y \neq x_{A'} - x_A$

$$\Delta y = y(x+\Delta x) - y(x)$$

$y(x)$ 是 x 处 偏离其平衡位置距离

在横波推导时，受力

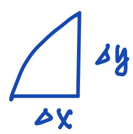
$$T_y = -\frac{\partial y}{\partial x} T \sim -\frac{\Delta y}{\Delta x} T$$

$$= -\frac{T}{\Delta x} \Delta y$$

以 $\frac{T}{\Delta x}$ 作为 K_{spring}，将会得到

和纵波一样的结果

看作耦合谐振子时，



以 Δx 段为研究对象

Δy 是形变，并非 $y(x+\Delta x)$

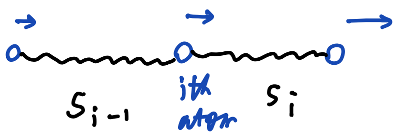
或 $y(x)$

张三慧先生的书用剪切应力来讨论，与此处一样。

前面的 Y（杨氏模量），张先生的书处理为

线应力

a: 原子间距　　$x_i \equiv ia$ （位置标记）



(i-1)-th spring ,　ith spring

第 i 个原子偏离平衡位置位移

$y_i \equiv y(x_i)$

第 i-th atom 受到 (i-1)-th spring 发生形变的力　　$-k_s(y_i - y_{i-1})$

deformation of (i-1)-th spring （弹簧）

与 the (i-1)th atom 受到此弹簧形变的力大小相等，方向相反。

$$\therefore F_i = \left(-k_s(y_i - y_{i-1})\right) - \left(-k_s(y_{i+1} - y_i)\right)$$

同样的弹簧，同样 k

$$= k_s\left[(y_{i+1} - y_i) - (y_i - y_{i-1})\right]$$

$$= k_s\left[\frac{\partial y}{\partial x}\Big|_{i+\frac{1}{2}} a - \frac{\partial y}{\partial x}\Big|_{i-\frac{1}{2}} a\right]$$

取中间位置

$$= k_s \frac{\partial^2 y}{\partial x^2} a^2$$

$$F_i = ma = m\frac{\partial^2 y}{\partial t^2} = k_s \frac{\partial^2 y}{\partial x^2} a^2 \Longrightarrow$$

$v = \frac{w}{k}$　　$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$　　$\omega^2 = k_s/m$

$a \sim \frac{1}{k}$

又见波动方程。