probability distribution

discrete $\rightarrow$ continuous

dice : 1, 2, 3, 4, 5, 6

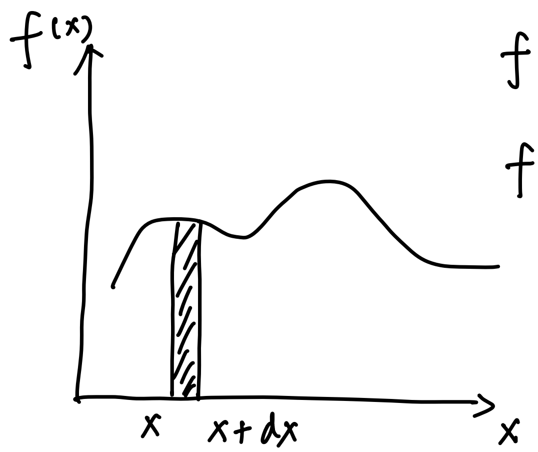
p_1 : 点数为1出现的概率

$= N_1/N$ (N : 总次数)

(N_1 : 点数为1出现次数)

同理 $p_2 \sim p_6$

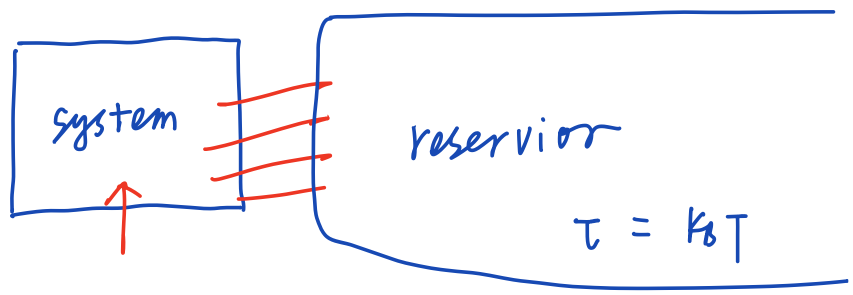
For continuous case.



$f(x)$: probability density

$f(x)dx$: probability of the regime $[x, x+dx)$

Q : what is the dimension of $f(x)$?



state s
with energy ε_s

For a statistical system in thermal equilibrium with a very large system (referred as reservior), the probability to find the system in state s is

$$p_s \propto e^{-\varepsilon_s/\tau}$$

$\rightarrow$ an interesting social phenomena

$$\int_0^\infty c\, e^{-AX}\, dX = 1$$

社会达成"热平衡"

Constraints: $\begin{cases} \text{人总数不变} \\ \text{薪水总额不变 (你多我少)} \end{cases}$

$\downarrow$ 平均薪水 τ 不变

$\begin{cases} m: \text{统计对象的薪水} \\ \tau: \text{平均薪水} \end{cases}$ $\tau = \langle m \rangle$

薪水为 m 人数占比 $p(m) = \frac{1}{\tau} e^{-m/\tau}$ (probability)

$$\int_0^m e^{-m/\tau} dm = \underline{\tau} \quad \text{(归一化系数)}$$

$$\frac{1}{\tau}\int_0^\infty e^{-m/\tau} dm = 1 \quad \left(\sum_i p_i = 1\right)$$

$$\tau = \int_0^\infty m\, p(m)\, dm$$

$$= \frac{1}{\tau}\int_0^\infty m\, e^{-m/\tau} dm$$

average salary

收入在 m 以下的占比：

$$C(m) \equiv \int_0^m p(m')\, dm'$$

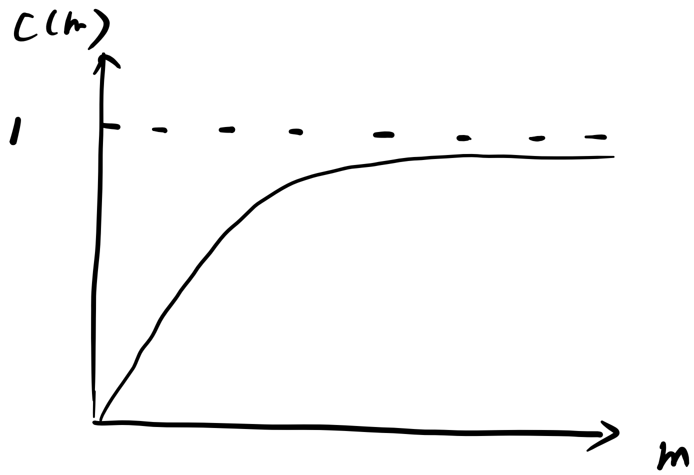
we have

$$1 = \int_0^\infty \frac{1}{\tau} e^{-m/\tau} dm \qquad = \int_0^m \frac{1}{\tau} e^{-m'/\tau} dm'$$

$$\tau = \int_0^\infty e^{-m/\tau} dm \qquad = 1 - e^{-m/\tau}$$

收入在 m 以下人群的平均薪水：$\dfrac{\int_0^m m' p(m')\, dm'}{C(m)}$

(人群基数不是全体)



$$1 - e^{-m_p/\tau} = \frac{1}{10} \quad \text{前 10\% poor}$$

$$m_p = \tau \log\left(\frac{10}{9}\right)$$

$$1 - e^{-m_r/\tau} = \frac{9}{10} \quad \text{前 10\% rich}$$

$$m_r = \tau \log 10$$

Disparity ratio $\equiv \dfrac{m_r}{m_p} = \dfrac{\tau \log 10}{\tau \log\left(\frac{10}{9}\right)} = 21.85$ (倍)

Salary : from poor to rich

Boltzmann distribution

two constraints:

$$\sum_i N_i = N$$

$$\sum_i N_i E_i = E$$

detailed balance (1872, Ludwig Boltzmann)

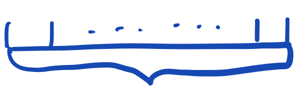
$$p\left((a,b) \to (a',b')\right) = p\left((a',b') \to (a,b)\right)$$

each elementary process is in equilibrium with its reverse process

$\downarrow$

$$\frac{N_a}{N}\frac{N_b}{N}\frac{2}{E_{a'}+E_{b'}} = \frac{N_{a'}}{N}\frac{N_{b'}}{N}\frac{2}{E_a+E_b}$$

(能级等间距)



$E_a + E_b$

choose one pair (two energy levels) (a',b') for transition $(a,b) \to (a',b')$

$$E_a + E_b = E_{a'} + E_{b'}$$

两个粒子发生碰撞，在总能量不变

的约束下 可变成任何允许的能量

detailed balance (即参与求和的任何能级)
细致平衡条件苛刻，达成 balance 可以是
(a, b)，(a', b') 作和的进出平衡。
"流守恒"

$$N_a N_b = N_{a'} N_{b'}$$

$$\log N_a + \log N_b = \log N_{a'} + \log N_{b'} \quad ①$$

$$E_a + E_b = E_{a'} + E_{b'} \quad ②$$

试探解 $\log N_a = AE_a + C_1$ 都能

满足 ① ②

$$N_a = C_2 e^{AE_a}$$

$$p_a = \frac{N_a}{N} = c e^{AE_a}$$

A, C 待定

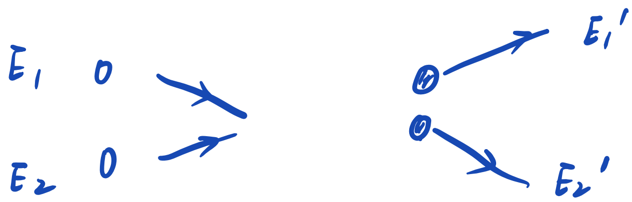
distribution

$$\sum_a p_a = 1 \Rightarrow \sum_a c e^{AE_a} = 1$$

答案：$\frac{1}{A} = -E/N$ (average energy)

Key physics: microscopic collisions

没有碰撞 → 走不到



Time reversal symmetry

$$\gamma = \gamma' \quad (\text{rate})$$

$$p(E_1)\,p(E_2)\,\gamma = p(E_1')\,p(E_2')\,\gamma'$$

$$E_1 + E_2 = E_1' + E_2' = E_T$$

$$p(E)\,p(E_T - E) = \text{const} \quad \text{for } \forall E$$

$$\log p(E) + \log p(E_T - E) = \text{const}$$

$$\left.\frac{d\log p}{dE}\right|_{E} = \left.\frac{d\log p}{dE}\right|_{E_T - E}$$

$$d\log p / dE = A \quad (\text{some constant})$$

$$\log p = AE + D$$

A, C

待定

$$p(E) = C e^{AE}$$

— Boltzmann distribution

Einstein use the similar way to handle photon → planck distribution

(quantum theory of emission and absorption of radiation)

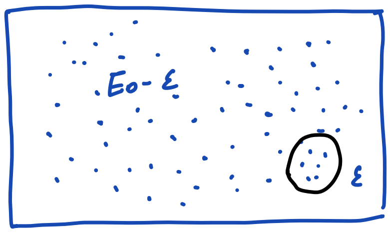
达成"热平衡"，所关心的系统将趋向满足约束条件下的"温度"。

如果讨论的对象是薪水，"温度"是平均薪水

~ 是分子运动， ~ 平均动能 (k_BT)

要确定前面所讨论到的常数 (A, C)：

等概率假设 + 熵最大



E_0 : total energy

a small system + the rest

$p \propto M(E_0-\varepsilon)$ (multiplicity)

$S \propto \ln M(E_0-\varepsilon)$, $S = k \ln M(E_0-\varepsilon)$

$S = S_0 - \varepsilon \frac{\partial S}{\partial E}\Big|_{E=E_0}$ ($\varepsilon \ll E_0$)

$$M(E_0-\varepsilon) = e^{S/k}$$

$$= \boxed{e^{S_0/k}}\, e^{-\varepsilon \frac{\partial S}{\partial E}\big|_{E=E_0}/k}$$

$$= \text{const}\ e^{-\varepsilon/kT}$$

$\frac{\partial S}{\partial E} = \frac{1}{T}$ (recall $dS = \frac{dQ}{T}$)

$$p \propto e^{-\varepsilon/k_B T}$$

addition of entropy : $S_1 \propto \ln M_1$, $S_2 \propto \ln M_2$

$M = M_1 M_2$, $S = S_1 + S_2$

refer to Lin's book p32 ~ 35

等概率原理：

对于平衡的孤立系统，系统的系统将等概率地分布于所有可能的状态。

将上图 ε (系统) + $E_0 - \varepsilon$ (环境)

const

合起来视为孤立系统 (E_0, V_0, N_0)

能量标记符号改为

$$E_1\ (\text{系统}) + E_2\ (\text{环境}) = E_0$$

$$V_1 + V_2 = V_0, \quad N_1 + N_2 = N_0$$

W_1, W_2: 系统，环境微观状态数

整个孤立系统的微观状态数 W_0 满足

$$W_0(E_0) = \sum_{E_1} W_1(E_1)\, W_2(E_0 - E_1)$$

求和遍历系统所有可能的能量取值

系统具有能量为 E_1 的概率

$$\rho(E_1) = \frac{W_1(E_1)\, W_2(E_0 - E_1)}{W_0(E_0)}$$

系统的能量平衡值

$$\bar{E}_1 = \sum_{E_1} \rho(E_1)\, E_1$$

系统是平衡的"宏观"系统，其能量标准差必然远小于能量平衡值，即 $\sqrt{\overline{\Delta E_1^2}} \ll \bar{E}_1$，因而 $\rho(\bar{E}_1)$ 应该在 $\bar{E}_1$ 处取尖锐的极大值：

$$\left.\frac{\partial \rho(E_1)}{\partial E_1}\right|_{\bar{E}_1} = 0$$

$$\left[\frac{\partial W_1(E_1)}{\partial E_1}\, W_2(E_0 - E_1) + W_1(E_1)\, \frac{\partial W_2(E_0 - E_1)}{\partial E_1}\right]_{\bar{E}_1} = 0$$

$$\left.\frac{\partial \ln W_1(E_1)}{\partial E_1}\right|_{\bar{E}} = \left.\frac{\partial \ln W_2(E_2)}{\partial E_2}\right|_{\bar{E}_2}$$

$$(\bar{E}_1 + \bar{E}_2 = E_0)$$

这一平衡定义了特征量

$$\beta(\bar{E}) = \left.\frac{\partial \ln(E)}{\partial E}\right|_{\bar{E}}$$

如果 $\beta_1 \neq \beta_2$

$$E_1 = \bar{E}_1 + \Delta, \quad E_2 = \bar{E}_2 - \Delta$$

那么在达成平衡的过程中将出现能量的定向转移，从而使 $\Delta \to 0$。联系热传导规律，

$$\beta(\bar{E}) = f(T)$$

（单调函数，一个平衡态对应一个温度）

利用热力学基本微分关系

$$dE = TdS - pdV + \mu dN$$

$$\begin{cases} \dfrac{1}{T} = \dfrac{\partial S}{\partial E} \\[2ex] \beta = \dfrac{\partial \ln W}{\partial E} \end{cases} \Longrightarrow \begin{cases} \dfrac{1}{T} \propto \beta(E) \\[2ex] S \propto \ln W + c \end{cases}$$

$$S_1 + S_2 = S \Longrightarrow c = 0$$

$$T = \frac{1}{k\beta}, \quad S = k\ln W$$

$\downarrow$

（墓碑上的公式）

Derivation of Boltzmann factor :

Lagrange multiplier

$p_m \propto e^{-m/\tau}$ maximize σ_I

$$\begin{cases} \sum_m p_m = 1 & (1) \\ \sum_m m p_m = \tau & (2) \end{cases}$$

$\sigma_I = -\sum_m p_m \log p_m$ (信息熵)

$$\sigma^* = \sigma_I + \lambda_1 \left(\sum_m p_m - 1 \right) + \lambda_2 \left(\sum_m m p_m - \tau \right)$$

Lagrange multipliers : λ_1 , λ_2

$\frac{\partial \sigma^*}{\partial \lambda_1} = 0 \longrightarrow 1^{st}$ constraint

$\frac{\partial \sigma^*}{\partial \lambda_2} = 0 \longrightarrow 2^{nd}$ constraint

$$\frac{\partial \sigma^*}{\partial p_m} = 0 \longrightarrow -\log p_m - p_m \cdot \frac{1}{p_m} + \lambda_1 + \lambda_2 m = 0$$

$$\log p_m = \lambda_1 + \lambda_2 m - 1$$

$$p_m = e^{\lambda_1 - 1} e^{\lambda_2 m}$$

$\downarrow$ const, rewriting $\frac{1}{Z}$

If only consider 1st constraint, i.e., $\lambda_2 = 0$

all states appear with equal probability

$\longrightarrow$ fundamental assumption!

Consider $\lambda_2 \neq 0 \longrightarrow \lambda_2 \rightarrow -\frac{1}{\tau}$

$$p_m \propto e^{-m/\tau}$$

entropy maximization, disorder maximization

with the energy constraint