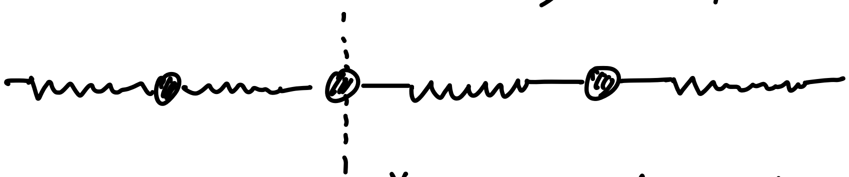
refer to prof. Hsin-Han Lin

sound waves propagating in solids and fluids & we have biological receptors for it.

Approximate the 1D solids as an array of coupled atoms mass M

$\rightarrow u_n(t)$: displacement for the nth atom



$X_n = na$ (equilibrium position)

$u(x,t) = u_n(t)$

$$M\ddot{u}_n = -K(u_n - u_{n-1}) + K(u_{n+1} - u_n)$$

$$= -Ka\frac{\partial u}{\partial x}\Big|_{n-\frac{1}{2}} + Ka\frac{\partial u}{\partial x}\Big|_{n+\frac{1}{2}}$$

$$= a^2 K \frac{\partial^2 u}{\partial x^2}\Big|_n$$

$$M\frac{\partial^2 u}{\partial t^2} = a^2 K \frac{\partial^2 u}{\partial x^2}$$

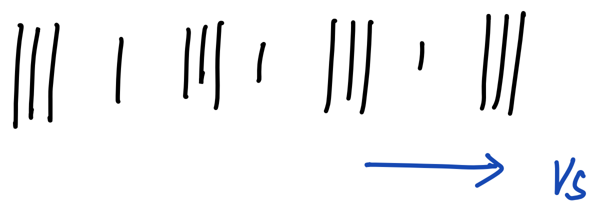
$$V = \sqrt{\frac{K}{M}}\, a$$

$w_0 = \sqrt{\frac{K}{M}}$ vibration

For stiff materials, the spring constant K is large $\rightarrow$ the sound speed is fast. Note that $V \sim 6000$ m/s in steel, much faster than $V \sim 330$ m/s in air.

Sound waves in an ideal gas

interesting observation



Q: what is the relevant parameters?

$$P,\ \rho,\ T,\ m,\ \ldots$$

what is your intuitive guess?

thermodynamic & microscopic quantities

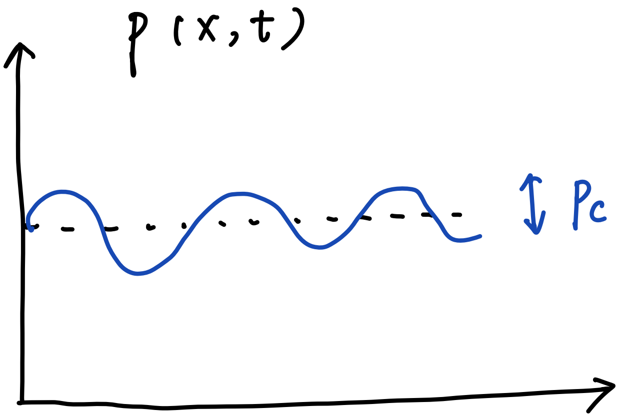
A rough picture for the sound wave helps:

a) gas molecules move and change the density.

b) change in density $\rightarrow$ change in pressure

c) pressure variations generate molecular motions.

The above picture suggests that we can identify p and ρ as the relevant parameters.



$$\begin{cases} p(x,t) = p_0 + p_c(x,t) \\ \rho(x,t) = \rho_0 + \rho_c(x,t) \end{cases}$$

(p_0, ρ_0 for equilibrium)

$$p = f(\rho)$$

It is reasonable to assume that $p_c \ll p_0$ and $\rho_c \ll \rho_0$ in usual sound waves

$$\begin{aligned} p_0 + p_c &= f(\rho_0 + \rho_c) \\ &= f(\rho_0) + f'(\rho_0)\rho_c + \frac{1}{2!} f''(\rho_0)\rho_c^2 + \cdots \end{aligned}$$

keeping the lowest non-vanishing term

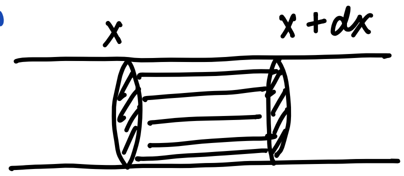
$$p_c(x,t) = K \rho_c(x,t)$$

$$K = f'(\rho_0) = \left(\frac{dp}{d\rho}\right)_0$$

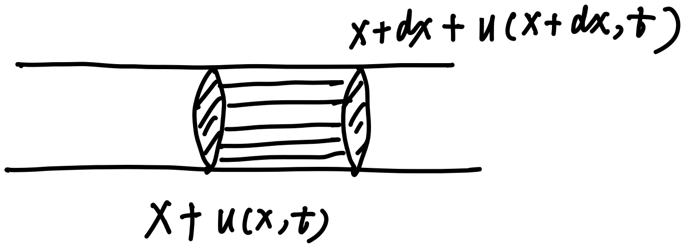
the generalized Hooke's law again:

$$p_c \propto \rho_c$$

the variation of the pressure $\propto$ ~ of the density demonstration



in equilibrium



$u(x,t)$: the displacement at position x and time t

From mass conservation in the tiny $(x \to x+dx)$ segment :

$$\rho_0 \, dx = \rho \, [\, x + dx + u(x+dx,t) - (x + u(x))\,]$$

$$= \rho \, [\, dx + \frac{\partial u}{\partial x} dx \,]$$

$$\rho_0 = \rho \left(1 + \frac{\partial u}{\partial x}\right) = (\rho_0 + \rho_c)\left(1 + \frac{\partial u}{\partial x}\right)$$

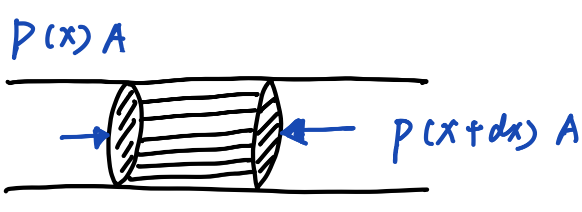
$$= \rho_0 + \rho_c + \rho_0 \frac{\partial u}{\partial x} + \cancel{\rho_c \frac{\partial u}{\partial x}} = 0$$

(smallness)

$$(\rho_c/\rho_0 \ll 1 \,, \quad \partial u / \partial x \ll 1)$$

Finally, we relate the density variation $\rho_c(x,t)$ to the spatial derivative of the scalar field $u(x,t)$

$$\rho_c(x,t) = -\rho_0 \frac{\partial u}{\partial x}$$



$$[\, -p(x+dx) + p(x) \,] \cancel{A} = \rho_0 \cancel{A} \, dx \, \frac{\partial^2 u}{\partial t^2}$$

(Newton 2nd law)

$$= -\frac{\partial p}{\partial x} dx = -\frac{\partial p_c}{\partial x} dx = -K \frac{\partial \rho_c}{\partial x} dx = K \rho_0 \frac{\partial^2 u}{\partial x^2} dx$$

$$K \cancel{\rho_0} \frac{\partial^2 u}{\partial x^2} \cancel{dx} = \cancel{\rho_0} \cancel{dx} \frac{\partial^2 u}{\partial t^2} \Rightarrow \frac{\partial^2 u}{\partial t^2} = K \frac{\partial^2 u}{\partial x^2}$$

wave equation $\quad V^2 = K = \left(\frac{dp}{d\rho}\right)_0$

similar to the derivation in the elastic wave

$$v^2 = K = \left(\frac{dp}{d\rho}\right)_0$$

dimension analysis:

$$p V_{vol} = N k_B T$$

$$p = \text{energy} / V_{vol}$$

$$\rho = \frac{m}{V_{vol}} = \text{energy} / v^2 V_{vol}$$

because the oscillation of the sound wave is quite fast, it can be viewed as adiabatic process (no heat in or out during oscillation)

$$p V_{vol}^{\gamma} = \text{const}$$

$$p = \text{const}\ \rho^{\gamma} = c \rho^{\gamma}$$

$$\left(\frac{dp}{d\rho}\right)_0 = \gamma c \rho^{\gamma-1} = \frac{\gamma p}{\rho}$$

$$v = \sqrt{\frac{\gamma p}{\rho}}$$

what about the process handled by the isothermal one ?

$$P V_{vol} = \text{Const}$$

$$P = c\rho$$

$$V = \sqrt{\frac{P}{\rho}}$$

you will make the same mistake as Newton did! (more than 10% molecules not involved ?)

so proud of you ^◡^

$$V = \sqrt{\frac{\gamma P}{\rho}}$$

due to $P V_{vol} = N k_B T$

$$P = \frac{N}{V_{vol}} k_B T$$

$$= \frac{N m}{m V_{vol}} = \frac{\rho V_{vol}}{m V_{vol}} k_B T$$

$$= \frac{\rho}{m} k_B T$$

(m : mass of a molecule)

$$\therefore \quad V = \sqrt{\frac{\gamma k_B T}{m}}$$

$$V_{rms} = \sqrt{\overline{V^2}} \quad (\text{root mean square})$$
$$= \sqrt{\frac{3 k_B T}{m}}$$

Newton's mechanics works again !
and explains the propagation of sound waves

k_B : 1.38×10^{-23} J / K

(dry air) T : 300 K (room temperature)

28.9647 g/mol $\quad m$: N_2 (78 vol%), O_2 (21 vol%)

The molar mass of a substance is defined as the mass of 1 mol of that substance expressed in grams per mole, and is equal to the mass of 6.022×10^{23} atoms, molecules

$$m_u = 1 \text{ g/mol}$$

$$V = \sqrt{\frac{1.4 \times 1.38 \times 10^{-23} \times 300}{28.9647 \times 10^{-3} / 6.02 \times 10^{23}}} \approx 347 \text{ m/s}$$