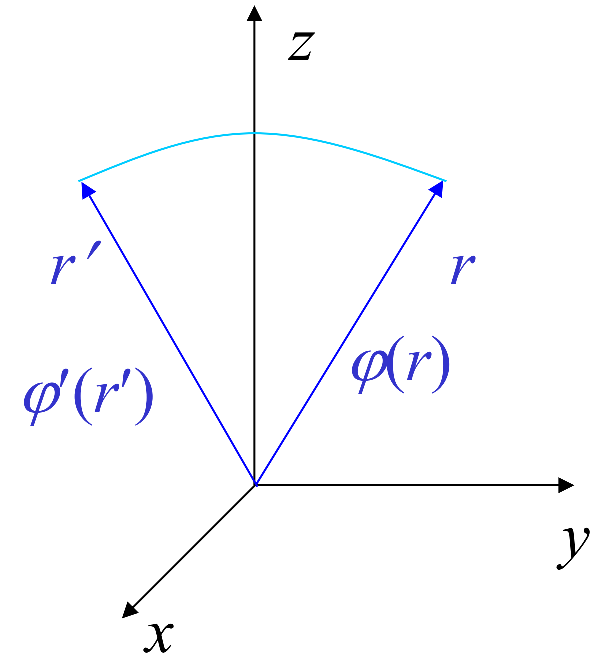
III. 表示的构造

函数的变换

矢量r在对称变化R作用下：$r'=Rr$

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{22} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$$r \to r', \varphi(r) \to \varphi'(r')$$

标量函数　比如温度场

$$\therefore \varphi'(r') = \varphi(r) = \varphi(R^{-1}r')$$

$$\therefore \varphi'(r) = \varphi(R^{-1}r)$$

令$P_R\varphi(r) = \varphi'(r)$

$$\therefore P_R\varphi(r) = \varphi(R^{-1}r)$$

P_R为作用于函数的算符 (函数变换算符)

如果R组成空间对称操作群，那么P_R在函数空间也构成一个群，且二者同构。

若 $G\{R,S,T,\cdots\},\quad RS=T$

则

$$P_R P_S \varphi(r) = P_R \varphi(S^{-1}r) = \varphi(S^{-1}R^{-1}r)$$

$$= \varphi[(RS)^{-1}r] = P_{RS}\varphi(r)$$

所以$\{P_R, P_S, P_T \ldots\}$也构成一个群

且$P_R \leftrightarrow R; P_S \leftrightarrow S; P_R P_S = P_{RS} \leftrightarrow RS; \cdots$

$\therefore$ {R}和{P_R}有相同的乘法表，同构。

标量函数的连续被动变换

Requirement: $\varphi(r) = \varphi'(r') = \varphi''(r'')$

连续变换关系： $r'' = Rr' = RSr$

$$\varphi''(r'') = \varphi'(R^{-1}r'')$$

变量替换： $r'' \to \bar{r}$

$$\varphi''(\bar{r}) = \varphi'(R^{-1}\bar{r}) \equiv \varphi'(\tilde{r}) \quad (\text{令 } \tilde{r} = R^{-1}\bar{r})$$

$$\varphi'(\tilde{r}) = \varphi(S^{-1}\tilde{r}) = \varphi(S^{-1}R^{-1}r)$$

$$\varphi''(r'') = \varphi(S^{-1}R^{-1}r)$$

$$\therefore P_R P_S \varphi(r) = \varphi(S^{-1}R^{-1}r)$$

群表示的确立

函数空间中取一组函数$\{\varphi_1(r), \varphi_2(r), \ldots \varphi_n(r)\}$作为基矢，基矢的数目等于空间的维数，则

$$P_R\varphi_\alpha(r) = \sum_\beta \varphi_\beta(r) D(R)_{\beta\alpha}$$

则$\{D(R)\}$就是群$\{P_R\}$和$\{R\}$的表示矩阵。

证明： $P_S\varphi_\beta(r)=\sum_\gamma \varphi_\gamma(r)D(S)_{\gamma\beta}$ $P_R\varphi_\alpha(r)=\sum_\beta \varphi_\beta(r)D(R)_{\beta\alpha}$

则： $P_SP_R\varphi_\alpha(r)=P_S\sum_\beta \varphi_\beta(r)D(R)_{\beta\alpha}=\sum_\beta\left[P_S\varphi_\beta(r)\right]D(R)_{\beta\alpha}$

$$=\sum_\beta\left[\sum_\gamma \varphi_\gamma(r)D(S)_{\gamma\beta}\right]D(R)_{\beta\alpha}=\sum_\gamma \varphi_\gamma(r)\left[\sum_\beta D(S)_{\gamma\beta}D(R)_{\beta\alpha}\right]$$

$$=\sum_\gamma \varphi_\gamma(r)\left[D(S)D(R)\right]_{\gamma\alpha}$$

另一方面： $P_SP_R\varphi_\alpha(r)=P_{SR}\varphi_\alpha(r)=\sum_\gamma \varphi_\gamma(r)D(SR)_{\gamma\alpha}$

$$\therefore D(S)D(R)=D(SR)$$

所以$\{D(R)\}$与$\{P_R\}$及$\{R\}$同构，是$\{P_R\}$及$\{R\}$的一个表示。

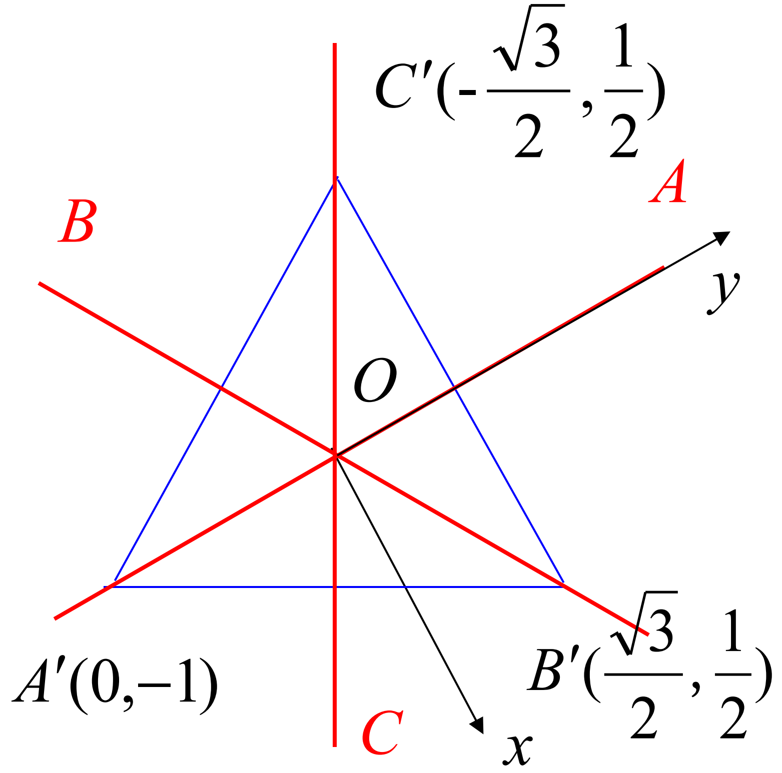
求群表示的方法之一：三角函数作基矢，(r,θ)为变量

例：求正三角形D_3群的表示

D_3群中的6个对称操作对r,θ的作用如下：

$E:\begin{cases} Er = r \\ E\theta = \theta \end{cases}$；$A:\begin{cases} Ar = r \\ A\theta = 180^\circ - \theta \end{cases}$；$B:\begin{cases} Br = r \\ B\theta = 60^\circ - \theta \end{cases}$；$C:\begin{cases} Cr = r \\ C\theta = 300^\circ - \theta \end{cases}$；

$D:\begin{cases} Dr = r \\ D\theta = \theta + 120^\circ \end{cases}$；$F:\begin{cases} Fr = r \\ F\theta = \theta - 120^\circ \end{cases}$；



取两个线性无关的基函数，
构造一个二维函数空间

$$\varphi_1(r,\theta) = \sin\theta$$

$$\varphi_2(r,\theta) = \cos\theta$$

求单位元素E的矩阵：

$$E:\begin{cases} Er = r \\ E\theta = \theta \end{cases}; \qquad \begin{aligned} \varphi_1(r,\theta) &= \sin\theta \\ \varphi_2(r,\theta) &= \cos\theta \end{aligned}$$

$$P_E\varphi_1 = \varphi_1 = \sin\theta = 1\sin\theta + 0\cos\theta$$

$$P_E\varphi_1 = \sum_i \varphi_i D_{i1} = \varphi_1 D_{11} + \varphi_2 D_{21} = \sin\theta D_{11} + \cos\theta D_{21}$$

$$P_E\varphi_2 = \varphi_2 = \cos\theta = 0\sin\theta + 1\cos\theta$$

$$P_E\varphi_2 = \sum_j \varphi_j D_{j2} = \varphi_1 D_{12} + \varphi_2 D_{22} = \sin\theta D_{12} + \cos\theta D_{22}$$

$$\therefore D(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

求元素A的矩阵：

$$A:\begin{cases} Ar = r \\ A\theta = 180^0 - \theta \end{cases} ; \qquad \begin{aligned} \varphi_1(r,\theta) &= \sin\theta \\ \varphi_2(r,\theta) &= \cos\theta \end{aligned}$$

$$P_A\varphi_1 = \varphi_1(A^{-1}\theta) = \varphi_1(A\theta) = \sin(180^0 - \theta) = \sin\theta + 0\cos\theta$$

$$P_A\varphi_1 = \sum_i \varphi_i D_{i1} = \varphi_1 D_{11} + \varphi_2 D_{21} = \sin\theta\, D_{11} + \cos\theta\, D_{21}$$

$$P_A\varphi_2 = \varphi_2(A^{-1}\theta) = \varphi_2(A\theta) = \cos(180^0 - \theta) = 0\sin\theta - 1\cos\theta$$

$$P_A\varphi_2 = \sum_j \varphi_j D_{j2} = \varphi_1 D_{12} + \varphi_2 D_{22} = \sin\theta\, D_{12} + \cos\theta\, D_{22}$$

$$\therefore D(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

求元素B,C的矩阵：

$$B:\begin{cases} Br = r \\ B\theta = 60^\circ - \theta \end{cases} \quad ; \quad C:\begin{cases} Cr = r \\ C\theta = 300^\circ - \theta \end{cases} \quad ; \quad \begin{matrix} \varphi_1(r,\theta) = \sin\theta \\ \varphi_2(r,\theta) = \cos\theta \end{matrix}$$

$$P_B\varphi_1 = \varphi_1(B^{-1}\theta) = \varphi_1(B\theta) = \sin(60^0 - \theta) = \sin 60^0 \cos\theta - \cos 60^0 \sin\theta$$

$$P_B\varphi_1 = \sum_i \varphi_i D_{i1} = \varphi_1 D_{11} + \varphi_2 D_{21} = \sin\theta\, D_{11} + \cos\theta\, D_{21}$$

$$P_B\varphi_2 = \varphi_2(B^{-1}\theta) = \varphi_2(B\theta) = \cos(60^0 - \theta) = \cos 60^0 \cos\theta + \sin 60^0 \sin\theta$$

$$P_B\varphi_2 = \sum_j \varphi_j D_{j2} = \varphi_1 D_{12} + \varphi_2 D_{22} = \sin\theta\, D_{12} + \cos\theta\, D_{22}$$

$$\therefore\ D(B) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$60^0 \rightarrow 300^0$

$$D(C) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

求元素D,F的矩阵：

D,F互逆

$$D:\begin{cases} Dr = r \\ D\theta = \theta + 120^0 \end{cases}; \quad F:\begin{cases} Fr = r \\ F\theta = \theta - 120^\circ \end{cases}$$

$$\varphi_1(r,\theta) = \sin\theta$$
$$\varphi_2(r,\theta) = \cos\theta$$

$$P_D\varphi_1 = \varphi_1(D^{-1}\theta) = \varphi_1(F\theta) = \sin(\theta - 120^0) = \sin\theta\cos 120^0 - \cos\theta\sin 120^0$$

$$P_D\varphi_1 = \sum_i \varphi_i D_{i1} = \varphi_1 D_{11} + \varphi_2 D_{21} = \sin\theta\, D_{11} + \cos\theta\, D_{21}$$

$$P_D\varphi_2 = \varphi_2(D^{-1}\theta) = \varphi_2(F\theta) = \cos(\theta - 120^0) = \sin 120^0 \sin\theta + \cos 120^0 \cos\theta$$

$$P_D\varphi_2 = \sum_j \varphi_j D_{j2} = \varphi_1 D_{12} + \varphi_2 D_{22} = \sin\theta\, D_{12} + \cos\theta\, D_{22}$$

$$D(D) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$120^0 \rightarrow 240^0$

$$D(F) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

所以正三角形对称群D_3的表示为：

$$D(E)=\begin{pmatrix}1&0\\0&1\end{pmatrix};D(A)=\begin{pmatrix}1&0\\0&-1\end{pmatrix};$$

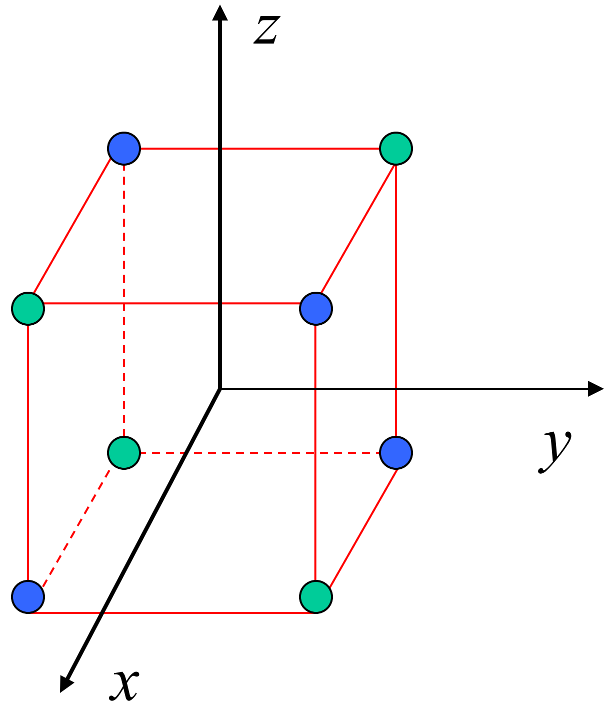
$$D(B)=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix};D(C)=\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix};$$

$$D(D)=\begin{pmatrix}-\frac{1}{2}&\frac{\sqrt{3}}{2}\\-\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix};D(F)=\begin{pmatrix}-\frac{1}{2}&-\frac{\sqrt{3}}{2}\\\frac{\sqrt{3}}{2}&-\frac{1}{2}\end{pmatrix};$$

求群表示的方法之二：以*x*, *y*, *z*为变量的函数空间

例：两种原子组成的四方晶体的对称操作所组成的群的表示

$$a = b \neq c$$



共有八个对称操作使晶格保持不变：

E：不动，

$C_2(z)$：绕z轴的2-度转动，

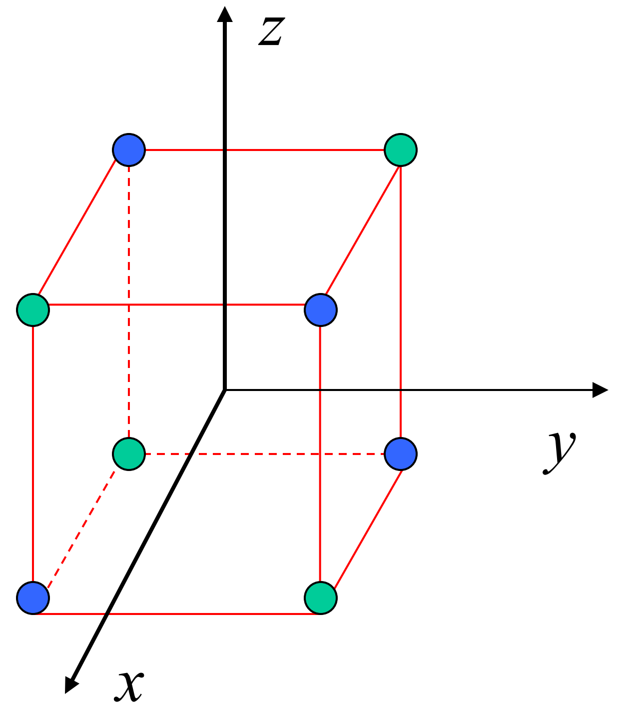
$C_2(x)$和$C_2(y)$：绕x轴和y轴的2-度转动，

σ_1和σ_2：关于对角平面(($y = x$和z轴）和（$y = -x$和z轴）)反射

iC_4和iC_4^{-1}：关于z轴4-度转动接着中心反演

选三个函数作为基矢建立一个3-维函数空间

$$\begin{cases}\varphi_1(\vec{r}) = x \\ \varphi_2(\vec{r}) = y \\ \varphi_3(\vec{r}) = z\end{cases}$$



用八个对称操作作用到三个基矢，例如：

自逆

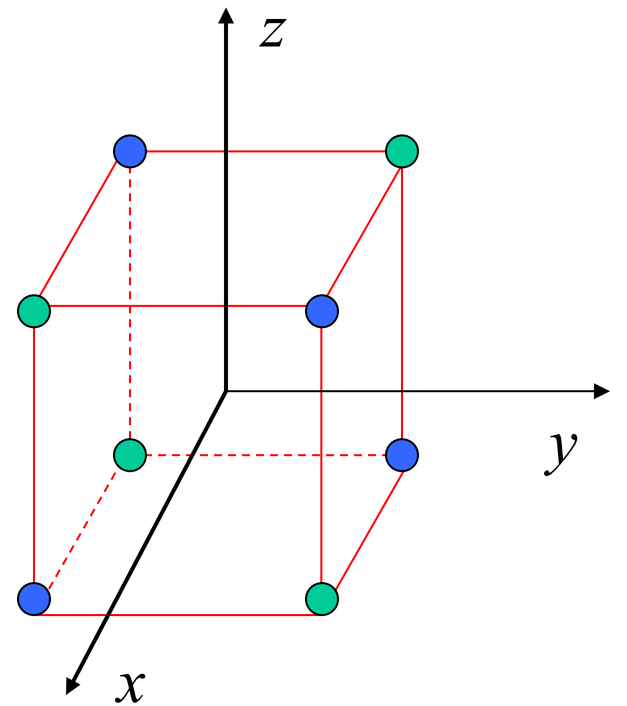
$$C_2(z)\varphi_1 = C_2(z)^{-1}\,[x] = [-x] = -\varphi_1$$
$$C_2(y)\varphi_1 = C_2(y)^{-1}x] = [-x] = -\varphi_1$$
$$C_2(x)\varphi_1 = C_2(x)^{-1}[x] = [+x] = \varphi_1$$
$$\sigma_1\varphi_1 = \sigma_1{}^{-1}[x] = [y] = \varphi_2$$
$$\sigma_2\varphi_1 = \sigma_2{}^{-1}[x] = [-y] = -\varphi_2$$

$$iC_4\varphi_1 = (iC_4)^{-1}[x] = iC_4{}^{-1}[x] = [-\mathrm{y}]$$
$$iC_4{}^{-1}\varphi_1 = iC_4[x] = iC_4[x] = [\mathrm{y}]$$

$$P_R\varphi(\vec{r}) = \varphi(R^{-1}\vec{r})$$

24次操作的结果:

	E	$C_2(z)$	$C_2(x)$	$C_2(y)$	σ_1	σ_2	iC_4	iC_4^{-1}
$\varphi_1=x$	φ_1	$-\varphi_1$	φ_1	$-\varphi_1$	φ_2	$-\varphi_2$	$-\varphi_2$	φ_2
$\varphi_2=y$	φ_2	$-\varphi_2$	$-\varphi_2$	φ_2	φ_1	$-\varphi_1$	φ_1	$-\varphi_1$
$\varphi_3=z$	φ_3	φ_3	$-\varphi_3$	$-\varphi_3$	φ_3	φ_3	$-\varphi_3$	$-\varphi_3$



根据函数变换的基本公式

$$P_R\varphi_i = \sum_j \varphi_j D_{ji}(R)\text{，即}(\varphi_1', \varphi_2', \varphi_3') = (\varphi_1, \varphi, \varphi_3)D(R)$$

则可以求得3-维表示。例如对于$D(C_2(z))$

$$(\varphi_1', \varphi_2', \varphi_3') = (-\varphi_1, -\varphi_2, \varphi_3) = (\varphi_1, \varphi_2, \varphi_3)\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

	E	$C_2(z)$	$C_2(x)$	$C_2(y)$	σ_1	σ_2	iC_4	iC_4^{-1}
$\varphi_1=x$	φ_1	$-\varphi_1$	φ_1	$-\varphi_1$	φ_2	$-\varphi_2$	$-\varphi_2$	φ_2
$\varphi_2=y$	φ_2	$-\varphi_2$	$-\varphi_2$	φ_2	φ_1	$-\varphi_1$	φ_1	$-\varphi_1$
$\varphi_3=z$	φ_3	φ_3	$-\varphi_3$	$-\varphi_3$	φ_3	φ_3	$-\varphi_3$	$-\varphi_3$

所以这个3-维表示为：

$$D(E)=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix};D(C_2(z))=\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix};D(C_2(x))=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix};D(C_2(y))=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix};$$

$$D(\sigma_1)=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix};D(\sigma_2)=\begin{pmatrix}0&-1&0\\-1&0&0\\0&0&1\end{pmatrix};D(iC_4)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-1\end{pmatrix};D(iC_4{}^{-1})=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&-1\end{pmatrix}$$

为方块对角矩阵，为右下角元素组成的1-维不可约表示$D^{(1)}$及左上角元素组成的2-维不可约表示$D^{(2)}$，则3-维可约表示可写成直和的形式：

$$D(a)=D^{(1)}(a)\oplus D^{(2)}(a)$$

所以用(x, y, z)为基矢求得的表示是可约化的。

特征标的性质

特征标 $\chi^j(R) = trD^j(R) = \sum_{\mu=1}^{m_j} D^j_{\mu\mu}(R)$

这是群元素$R \in G$在第 j 套表示中的特征标。

1）正交定理

$$\frac{1}{g}\sum_{R\in G} D^i_{\mu\rho}(R)^* D^j_{\nu\lambda}(R) = \frac{1}{m_j}\delta_{ij}\delta_{\mu\nu}\delta_{\rho\lambda}$$

取$\mu = \rho$，$\nu = \lambda$，并对μ和ν求和得：

$$\frac{1}{g}\sum_{R\in G}\sum_{\mu} D^i_{\mu\mu}(R)^* \cdot \sum_{\nu} D^j_{\nu\nu}(R) = \frac{1}{m_j}\underbrace{\sum_{\mu}\sum_{\nu}}_{m_j}\delta_{\mu\nu}\delta_{\mu\nu}\delta_{ij}$$

即 $\sum_{R\in G}\chi^i(R)^*\chi^j(R) = g\delta_{ij}$

正三角形D_3的不可约表示：

	E	A	B	C	D	F
D^1	1	1	1	1	1	1
D^2	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

$$\sum_{R\in G} \chi^1(R)^* \chi^2(R) = 1\times 2 + 1\times 0 + 1\times 0 + 1\times 0 + 1\times(-1) + 1\times(-1) = 0 = g\delta_{12}$$

$$\sum_{R\in G} \chi^2(R)^* \chi^2(R) = 2\times 2 + 0 + 0 + 0 + (-1)\times(-1) + (-1)\times(-1) = 6 = g\delta_{22}$$

2）特征标是相似变换下的不变量，故有：

i）等价表示具有相同的特征标

ii）同一共轭元素类中诸元素具有相同的特征标

（实际上，特征标是类的函数）

证：若S同R共轭，则$S = T^{-1}RT$，T，S，$R \in G$

则 $$D(S) = D(T^{-1}RT) = D(T^{-1})D(R)D(T)$$

$$\chi(S) = \chi(T^{-1}RT) = tr\{D(T^{-1})D(R)D(T)\} = tr\{D(T)D(T^{-1})D(R)\}$$

$$= tr\{D(E)D(R)\} = trD(R) = \chi(R)$$

设群G有S个类，第α类含h_α个元素（群元素个数），则第二正交关系改写为：

$$\sum_{\alpha=1}^{s} h_\alpha \chi_\alpha^{i*} \chi_\alpha^{j} = g\delta_{ij}$$

3) 一个可约表示的特征标，等于约化后各不可约表示的特征标之和

$$D = \sum_j \oplus a_j D^j(R)$$

$$\chi(R) = \sum_j \chi^j(R) a_j$$

第j个不可约表示在可约表示中出现的次数, 称为**约化系数**

$$\sum_{R\in G} \chi^i(R)^* \chi(R) = \sum_{R\in G} \sum_j \chi^i(R)^* \chi^j(R) a_j$$

$$= \sum_j \left[\sum_{R\in G} \chi^i(R)^* \chi^j(R) \right] a_j$$

$$= \sum_j \delta_{ij} g\, a_j = g a_i$$

类C的群元数目

$$a_i = \frac{1}{g} \sum_{R\in G} \chi^i(R)^* \chi(R) = \frac{1}{g} \sum_C h_C\, \chi^i(C)^* \chi(C)$$

相关推论：

约化系数$\boldsymbol{a_i}$被唯一确定

(a) 如果给定这个表示的特征标系与某一个不可约表示的特征标系 χ_G^i 完全相同，那么，给定的表示是不可约表示，而且与 D_G^i 等价。

(b) 如果给定的表示的特征标系与任何一个不可约表示的特征标系都不同，那么这个表示肯定是可约表示。利用约化系数的公式可将其约化。

4）不可约表示的判据

$\sum_{R\in G}\chi(R)^*\chi(R) = g$ 或 $\sum_C h_C\chi(C)^*\chi(C) = g$

证明： 已知

$$\chi(R) = \sum_j \chi^j(R)a_j$$

$$\chi(R)^* = \sum_j \chi^j(R)^* a_j^*$$

两式相乘

$$\frac{1}{g}\sum_{R\in G}\chi(R)^*\chi(R) = \frac{1}{g}\sum_{R\in G}\sum_i \chi^i(R)^* a_i^* \sum_j \chi^j(R)a_j$$

$$= \frac{1}{g}\sum_{ij} a_i^* a_j \sum_{R\in G}\chi^i(R)^*\chi^j(R)$$

$$= \frac{1}{g}\sum_{ij} a_i^* a_j \delta_{ij} g$$

$$= \sum_i |a_i|^2$$

只有某个$a_i = 1$，成为不可约表示D_G^i本身，其它系数都为0

用判据检测上述表示的不可约性 $\sum_{R\in G}\chi(R)^*\chi(R)=g$

例 1

	E	A	B	C	D	F
D^1	1	1	1	1	1	1
D^2	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$

例 2

$$D(E)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix};D(C_2(z))=\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix};D(C_2(x))=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix};D(C_2(y))=\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix};$$

$$D(\sigma_1)=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix};D(\sigma_2)=\begin{pmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix};D(iC_4)=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix};D(iC_4^{-1})=\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$