Hermann Grassmann
15 Apr 1809 – 26 Sep 1877

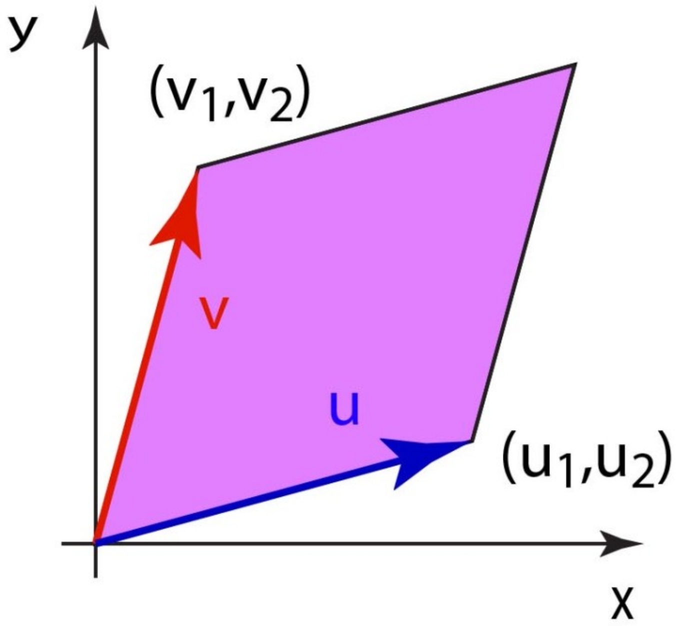
Parallelogramm

und derselbe ist. Jene zwei Strecken bildeten dann die Seiten des Spathecks, und diese drei die Kanten des Spathes, und zwar nahmen wir dort die Strecke, durch deren Bewegung das Spatheck entstand, als ersten, die die Bewegung messende als zweiten Faktor an, und setzten zwei Spathecke als gleich bezeichnet, wenn der zweite Faktor vom ersten aus betrachtet nach derselben Seite hin liegt, wenn nach entgegengesetzter, als entgegengesetzt bezeichnet. Hierin liegt schon das Gesetz, dass

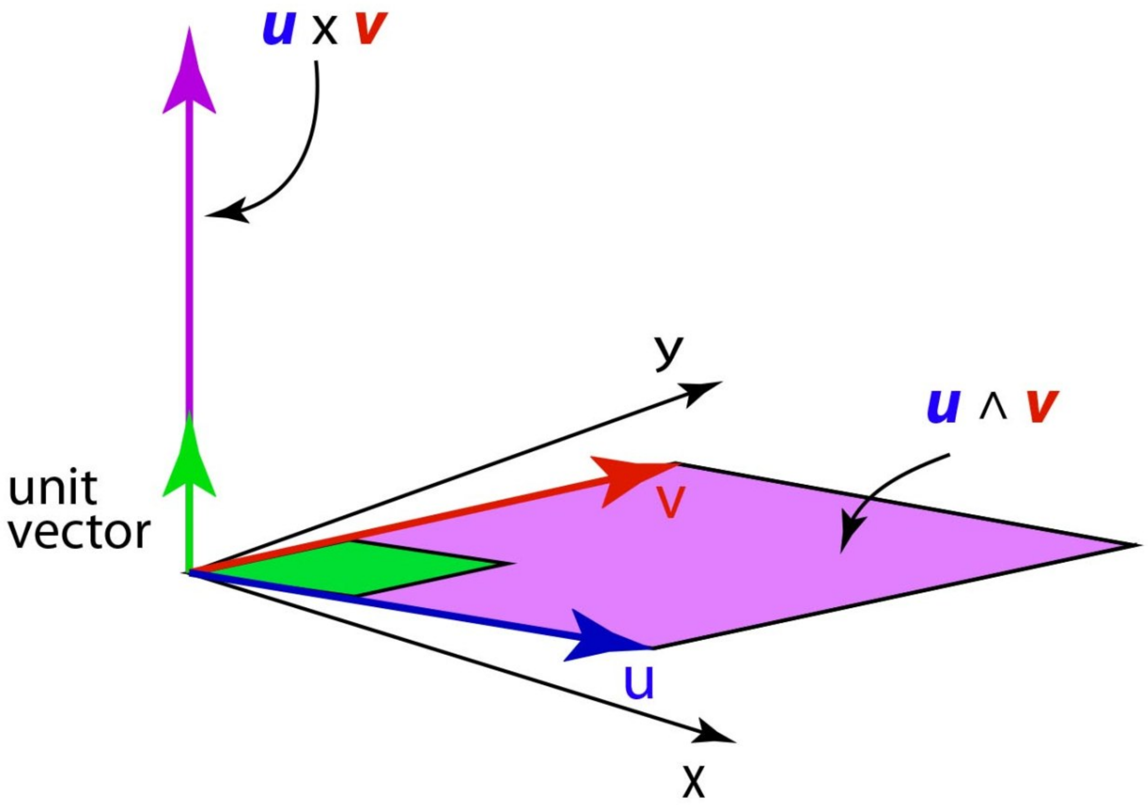
$$a . b = - b . a$$

ist; denn wenn b von a aus betrachtet nach links liegt, so muss a von b aus betrachtet nach rechts hin liegen und umgekehrt. Allein um diesem Vertauschungsgesetz, was die hier aufgestellte Multiplikation auf eine so auffallende Weise von der gewöhnlichen ausscheidet, eine noch anschaulichere Basis zu geben, will ich auch jenes allgemeinere Zeichengesetz, von dem dieses eine specielle Folgerung enthält, auf geometrische Weise ableiten. Zuerst ist

Area 2-form



$$Area = u_1v_2 - u_2v_1$$



$$x \wedge y = -y \wedge x$$

$$x \wedge x = 0$$

1-form $\quad \alpha = ax + by + cz$

$$\alpha = ax + by + cz$$
$$\beta = dx + ey + fz$$
$$\alpha \wedge \beta = (ax + by + cz) \wedge (dx + ey + fz)$$
$$= aex \wedge y + afx \wedge z + bdy \wedge x + bfy \wedge z + cdz \wedge x + cez \wedge y$$
$$= (ae - bd) x \wedge y + (af - cd) x \wedge z + (bf - ce) y \wedge z$$

$$y \wedge z \rightarrow \hat{x}$$
$$z \wedge x \rightarrow \hat{y}$$
$$x \wedge y \rightarrow \hat{z}$$

It is mapping in 3D, not equality!

$$\alpha \wedge \beta = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a & b & c \\ d & e & f \end{vmatrix} = \hat{x}(bf - ce) + \hat{y}(cd - af) + \hat{z}(ae - bd)$$

$$= y \wedge z(bf - ce) + z \wedge x(cd - af) + x \wedge y(ae - bd)$$

$$\vec{u}\wedge\vec{v}=\left(e_2\wedge e_3\right)\left(u^2v^3-u^3v^2\right)+\left(e_1\wedge e_2\right)\left(u^1v^2-u^2v^1\right)+\left(e_1\wedge e_3\right)\left(u^1v^3-u^3v^1\right)$$

associativity

$$\left(\alpha\wedge\beta\right)\wedge\gamma=\alpha\wedge\left(\beta\wedge\gamma\right)=\alpha\wedge\beta\wedge\gamma$$

The triple cross product is not associate

In 3-space

$$\vec{u} \wedge \vec{v} = \left(e_2 \wedge e_3\right)\left(u^2 v^3 - u^3 v^2\right) + \left(e_1 \wedge e_2\right)\left(u^1 v^2 - u^2 v^1\right) + \left(e_1 \wedge e_3\right)\left(u^1 v^3 - u^3 v^1\right)$$

Generalize to high dimensions

$$\vec{u} \wedge \vec{v} = \left(\hat{e}_a \wedge \hat{e}_b\right)\left(u^a v^b - u^b v^a\right)$$

Multivector：

$$\delta = a + bx + cy + dz + ex \wedge y + fy \wedge z + gz \wedge x + hx \wedge y \wedge z$$

$2^3 = 8$ elements: one scalar; three 1-forms; three 2-forms; one 3-forms

Exponentially increasing

$$\alpha = a^1 x + a^2 y + a^3 z$$

$$\beta = b^1 x + b^2 y + b^3 z$$

$$\gamma = g^1 x + g^2 y + g^3 z$$

$$\alpha \wedge \beta \wedge \gamma = \left(a^1 x + a^2 y + a^3 z\right) \wedge \left(b^1 x + b^2 y + b^3 z\right) \wedge \left(g^1 x + g^2 y + g^3 z\right)$$

$$= \left(a^1 b^2 g^3 + a^2 b^3 g^1 + a^3 b^1 g^2 - a^1 b^3 g^2 - a^2 b^1 g^3 - a^3 b^2 g^1\right) x \wedge y \wedge z$$

$$= \varepsilon_{ijk} a^i b^j g^k e_1 \wedge e_2 \wedge e_3$$

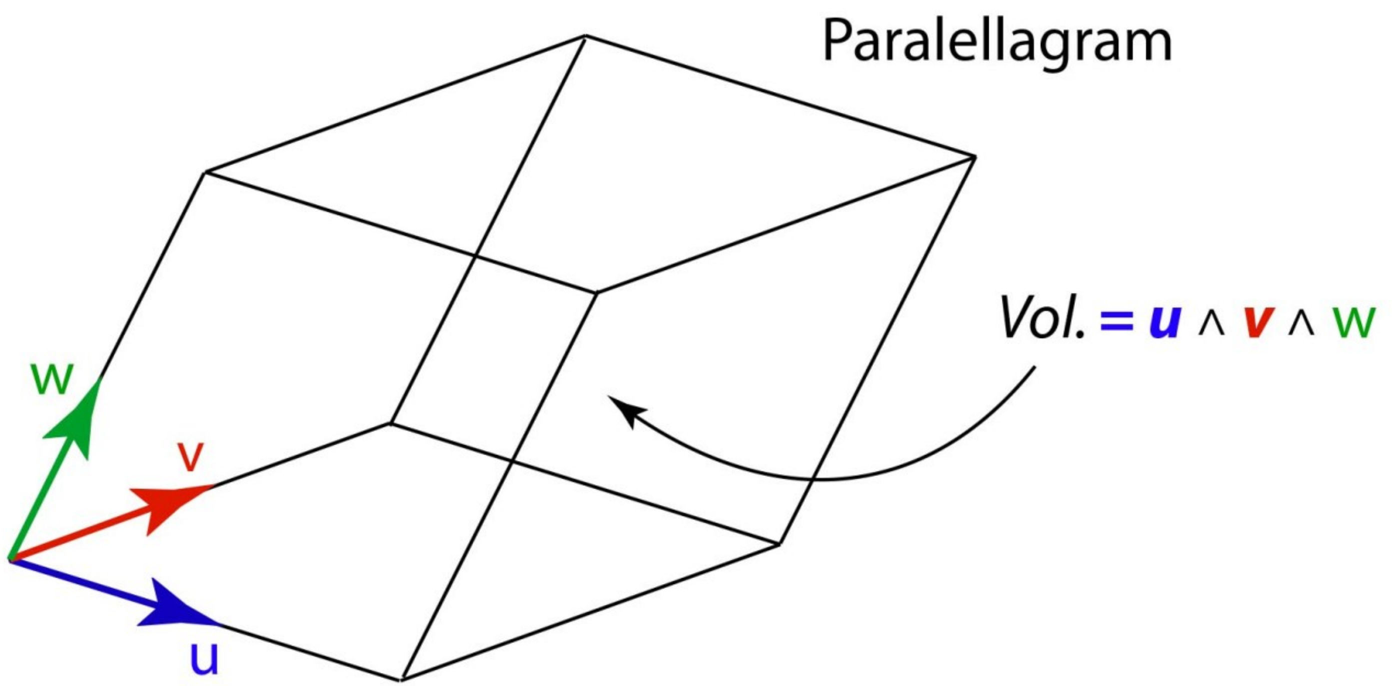
$$\varepsilon_{ijk} = \begin{cases} +1 & even - permutation \\ -1 & odd - permutation \end{cases}$$

Tullio Levi-Civita

29 Mar 1873 – 29 Dec 1941

Volume 3-form

$$\begin{aligned} Vol. &= \vec{u}\cdot(\vec{v}\times\vec{w}) \\ &= \vec{v}\cdot(\vec{w}\times\vec{u}) \\ &= \vec{w}\cdot(\vec{u}\times\vec{v}) \end{aligned}$$



$$Vol. = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} = u_1(v_2w_3 - w_2v_3) + v_1(w_2u_3 - w_3u_1) + w_1(u_2v_3 - v_2u_3)$$

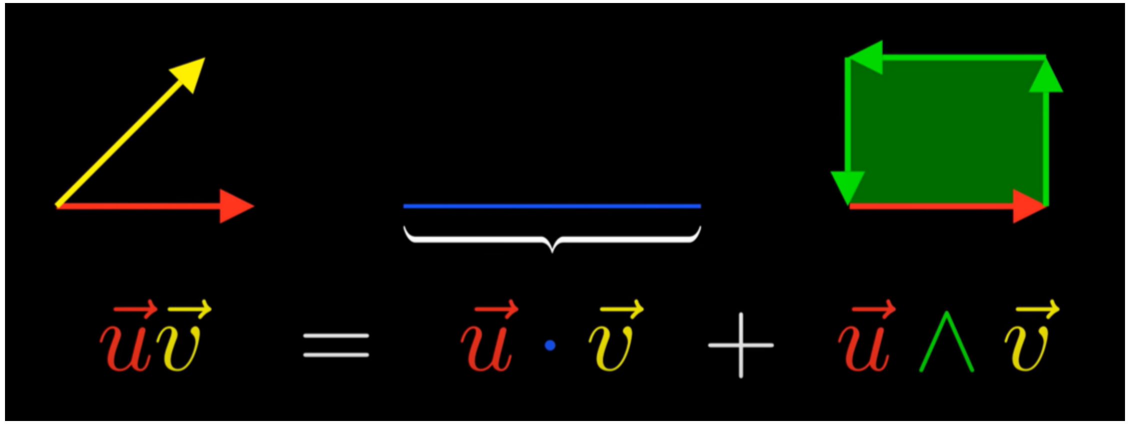
Modern view – Geometric Algebra

- What information will we lose if putting two vectors (**bivector**) together?
- $\vec{u} \cdot \vec{v}$: scalar, inner product : projection
- $\vec{u} \wedge \vec{v}$: **bivector (in some sense, pseudo-vector in 3D)** :

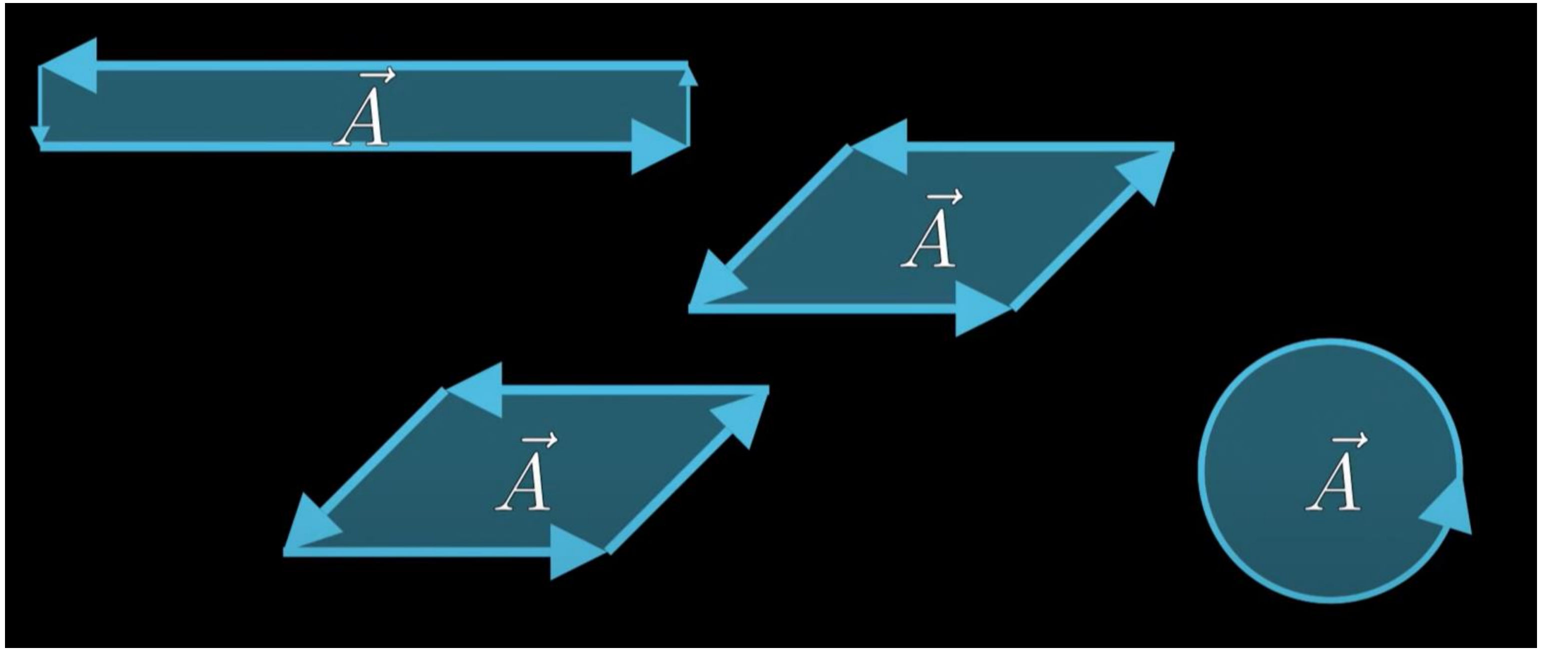
  right-hand grip rule

Combine : $\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$

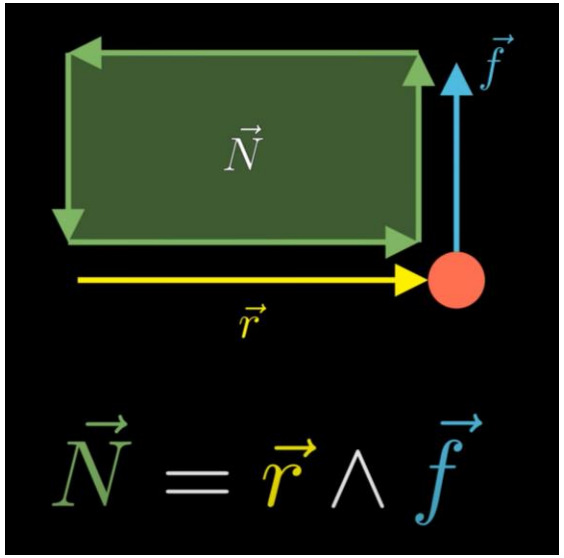
$\vec{u}^2 = \|\vec{u}\|^2$

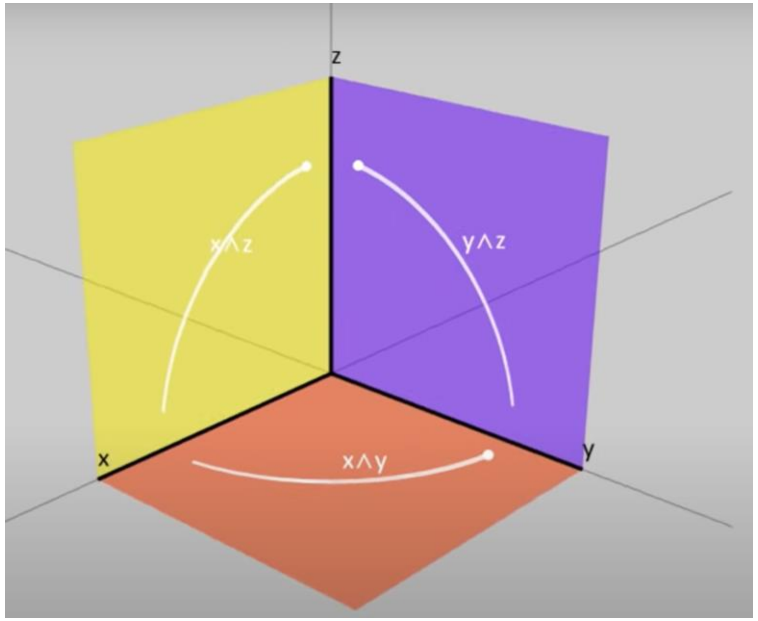


The degeneracy of bivectors:
the same magnitude (area), the same orientation

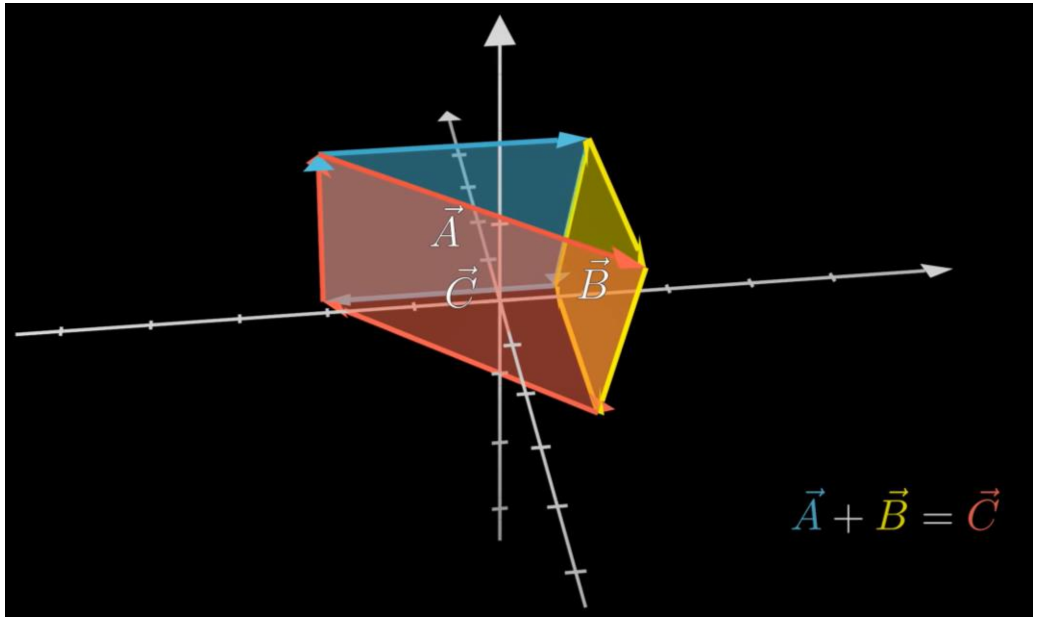



Rotations happen in 2D planes
z
x
y
This old lady is spinning wheel in the xz plane, perpendicular to the y axis.


$\vec{N}$
$\vec{f}$
$\vec{r}$
$\vec{N} = \vec{r} \wedge \vec{f}$


z
x∧z
y∧z
x
x∧y
y

How to add ?


$\vec{A}$
$\vec{B}$
$\vec{C}$
$\vec{A} + \vec{B} = \vec{C}$

outer product vs cross product

What do we have?

$$\vec{u}^2 = \|\vec{u}\|^2$$

$$\vec{u}\wedge\vec{v} = -\vec{v}\wedge\vec{u}$$

We need one more:

$$i \equiv \hat{x}\hat{y}\hat{z}$$

$$\vec{u}\wedge\vec{v} = i\vec{u} \times \vec{v}$$

$$\vec{u} = a_1\hat{x} + b_1\hat{y} + c_1\hat{z}$$

$$\vec{v} = a_2\hat{x} + b_2\hat{y} + c_2\hat{z}$$

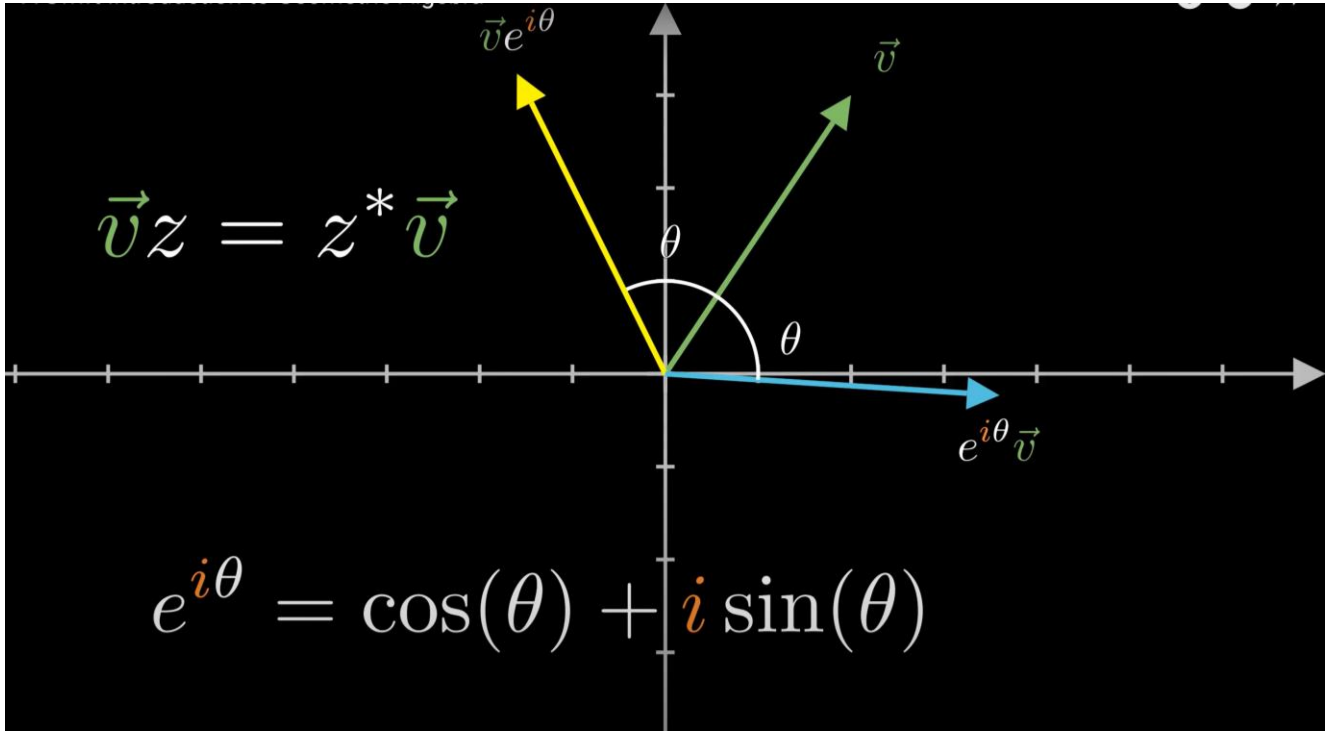
$$\vec{u}\vec{v} = a_1a_2 + b_1b_2 + c_1c_2 + (a_1b_1 - b_1a_2)\hat{x}\hat{y} + (b_1c_2 - c_1b_2)\hat{y}\hat{z} + (a_1c_2 - c_1a_2)\hat{x}\hat{z}$$

$$\vec{u}\cdot\vec{v} = a_1a_2 + b_1b_2 + c_1c_2$$

$$\vec{u}\wedge\vec{v} = (a_1b_2 - b_1a_2)\hat{x}\hat{y} + (b_1c_2 - c_1b_2)\hat{y}\hat{z} + (a_1c_2 - c_1a_2)\hat{x}\hat{z}$$

$$\vec{u}\times\vec{v} = (a_1b_2 - b_1a_2)\hat{z} + (b_1c_2 - c_1b_2)\hat{x} - (a_1c_2 - c_1a_2)\hat{y}$$

$$\vec{u}\wedge\vec{v} = i\vec{u}\times\vec{v}$$



In **2D**, $a\hat{x}\hat{y} \equiv ai$

$$\vec{u}i = -i\vec{u}$$

$$i^2 = \hat{x}\hat{y}\hat{x}\hat{y} = -\hat{x}\hat{x}\hat{y}\hat{y} = -1$$

$$\vec{u}\,\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\theta + \|\vec{u}\|\|\vec{v}\|\sin\theta\; i = \|\vec{u}\|\|\vec{v}\|e^{i\theta}$$

Pseudo-vectors

Pseudo-scalars

In **3D**, $a\hat{x}\hat{y}\hat{z} \equiv ai$

$$\vec{u}i = i\vec{u}$$

$$Ai = iA$$

$$(A\text{: } multivector)$$

$$i^2 = \hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = -1$$

$\hat{x}i = \hat{x}\hat{x}\hat{y}\hat{z} = \hat{y}\hat{z}$ (bivector)

$$\vec{L}, \vec{B}$$

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = i\vec{a}\wedge\vec{b}\wedge\vec{c}$$

$\boldsymbol{\Phi_B}$ **: trivector**

Chirality is so natural in the algebra.

3D is so special: the same size for vector and pseudovector:

$\{e_1, e_2, e_3\}$ $\{e_1e_2, e_2e_3, e_1e_3\}$

furthermore

- $e_i e_i = 1$, $e_i e_j = -e_j e_i$

- $\hat{x}\hat{y} = \hat{x}\hat{y}\hat{z}\hat{z} = i\hat{z}$
- $\hat{y}\hat{z} = \hat{x}\hat{x}\hat{y}\hat{z} = \hat{x}i$
- $\hat{z}\hat{x} = \hat{z}\hat{x}\hat{y}\hat{y} = i\hat{y}$

- $\hat{x}^2 = \hat{y}^2 = \hat{z}^2 = 1$

It is just **Pauli matrix**

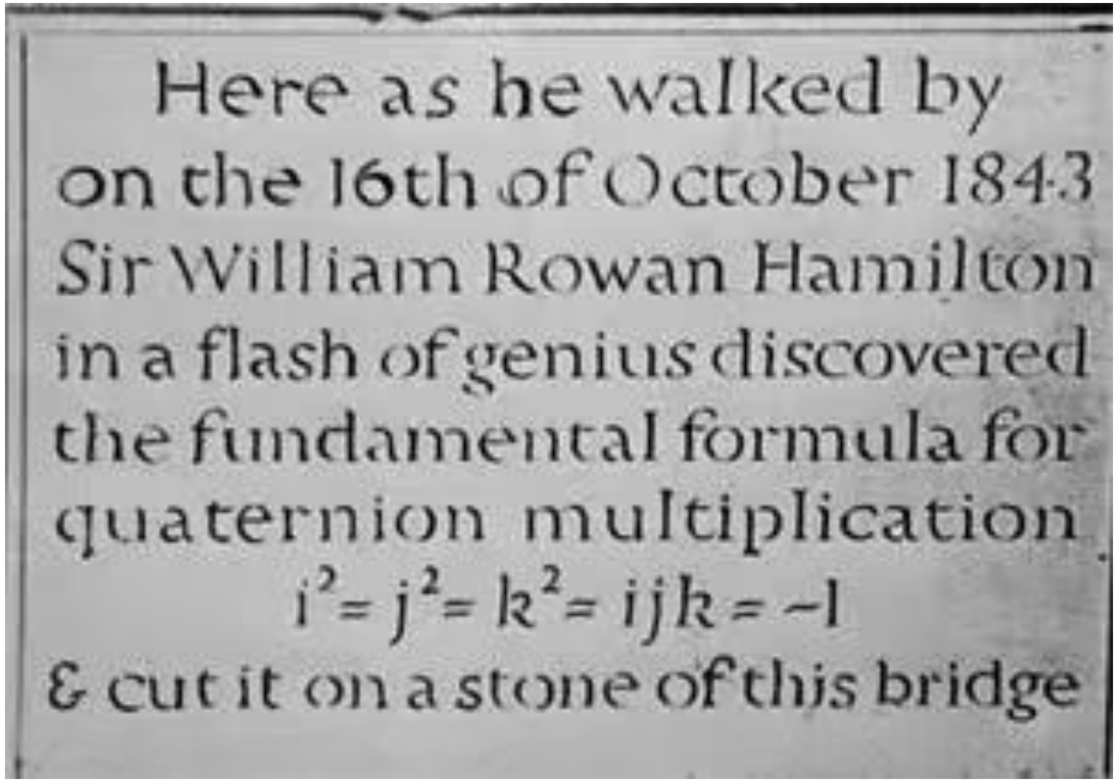
$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

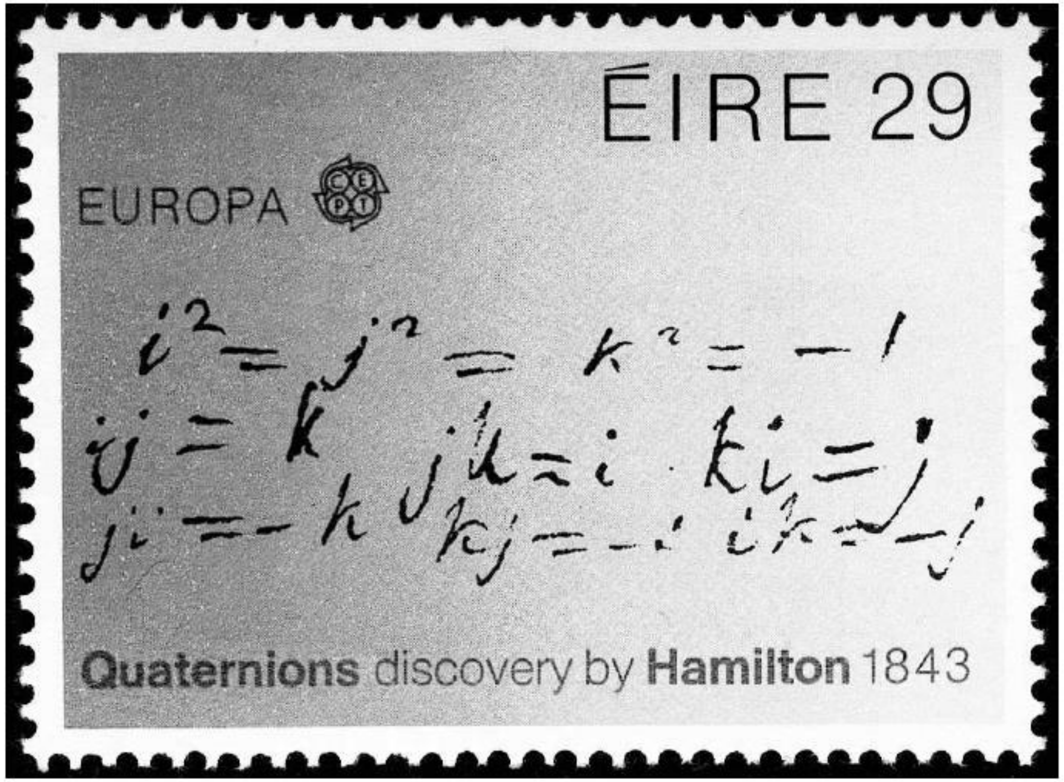
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In 3D

$$(\hat{x}\hat{y})^2 = (\hat{y}\hat{z})^2 = (\hat{x}\hat{z})^2 = -1$$
$$\hat{x}\hat{y}\hat{y}\hat{z}\hat{x}\hat{z} = -1$$





1843.10.16, 都柏林布鲁穆桥

哈密顿--- 刻在桥上的公式

The quarternionic group ——8阶群

group elements: $\{1, -1, i, -i, j, -j, k, -k\}$

Vs 复数：$a + bi$，有两个单位 $(1, i)$，—— 二元数。

定义四元数：$q = a + bi + cj + dk$

且规定：

$$1^2 = 1, (-1)^2 = 1,$$
$$i^2 = j^2 = k^2 = ijk = -1,$$

则有

$$ij = -ji = k$$
$$jk = -kj = i,$$
$$ki = -ik = j$$

The eight-element quarternionic group is to quartenionic numbers as the four-element cyclic group Z_4 is to complex numbers

Geometric Algebra (in 3D VGA):

$$a + b\hat{x} + c\hat{y} + d\hat{z} + e\hat{x}\hat{y} + f\hat{y}\hat{z} + g\hat{x}\hat{z} + h\hat{x}\hat{y}\hat{z}$$

Vectors: $a\hat{x} + b\hat{y} + c\hat{z}$

Complex numbers: $a + bi$

Quaternions: $a + bi + cj + dk$

Exterior Algebra: $A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}$

Vector Algebra: $\vec{u} \times \vec{v} = -i\vec{u} \wedge \vec{v}$

GA **see** Maxwell equations:

max well equations :

$$F = \vec{E} + ic\vec{B}$$

$$\nabla F = J / c\varepsilon_0$$

$$J = (c\rho, \vec{J})$$

$$\nabla = \frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}$$

$$\nabla F = \left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\nabla}\right)\left(\vec{E} + ic\vec{B}\right)$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + i\frac{\partial}{\partial t}\vec{B} + \vec{\nabla}\vec{E} + ic\vec{\nabla}\vec{B}$$

$$\vec{\nabla}\vec{E} = \vec{\nabla}\cdot\vec{E} + \vec{\nabla}\wedge\vec{E}$$

$$\vec{\nabla}\vec{B} = \vec{\nabla}\cdot\vec{B} + \vec{\nabla}\wedge\vec{B}$$

↓

$$\vec{\nabla}\cdot\vec{E} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} + ic\vec{\nabla}\wedge\vec{B} + \vec{\nabla}\wedge\vec{E} + i\frac{\partial\vec{B}}{\partial t} + ic\vec{\nabla}\cdot\vec{B}$$

scalar, vector, bivector, trivector

$$= \frac{\rho}{\varepsilon_0} - \frac{\vec{J}}{c\varepsilon_0}$$

$$\therefore \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \quad ①$$

$$\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + ic\vec{\nabla}\wedge\vec{B} = -\frac{\vec{J}}{c\varepsilon_0}$$

↓

$$\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + i\vec{\nabla}\wedge\vec{B} = -\frac{\vec{J}}{c^2\varepsilon_0}$$

$$\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} + i\vec{\nabla}\wedge\vec{B} = -\mu_0\vec{J}$$

$$i\vec{\nabla}\wedge\vec{B} = i\,i\,\vec{\nabla}\times\vec{B}$$

↓

$$\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right) \quad ②$$

$$\vec{\nabla}\wedge\vec{E} + i\frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad ③$$

$$ic\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0 \quad ④$$

- A unified mathematical language for the whole of physics

---------David Hestenes

The most powerful and general language available for the development of mathematical physics

----------------- Stephen Gull

Anthony Lasenby

and Chris Doran

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G(3)

trivector/ pseudoscalar	$e_1e_2e_3$	grade-3
bivector/ pseudovector	e_1e_2 e_2e_3 e_3e_1	grade-2
vectors	e_1 e_2 e_3	grade-1
scalars	1	grade-0