$N \times N$ $matrices$

$SU(N)$ $SO(N)$

To solve

$U^{\dagger}U = 1,$
$detU = 1$

$O^{t}O = E,$
$\det O = 1$

A.Zee: group theory in a nutshell for physicists

- **I.3 Rotations and the Notion of Lie Algebra**
- Do what you have to do a little bit at a time
- An infinitesimal angle θ (near the identity)
- $R(\theta) \approx I + A$

Norwegian physicist Marius Sophus Lie (1842-1899)

Orthogonal matrix $R^T R = 1$

$R^T R \approx (I + A)(I + A^T) \approx (I + A + A^T) = I \rightarrow$

$A = -A^T$ (antisymmetric) $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow A = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

⇓

generator of the rotation group: $R = I + \theta \mathcal{J} + O(\theta^2) = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix} + O(\theta^2)$

- $SO(2)$: $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \cos\theta\, I + \sin\theta\, \mathcal{J}$

$\theta \to 0$

$$\begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}$$

$$R(\theta) = \lim_{N\to\infty}\left(R\left(\frac{\theta}{N}\right)\right)^N = \lim_{N\to\infty}\left(1+\frac{\theta\mathcal{J}}{N}\right)^N = \lim_{N\to\infty}\left(e^{\frac{\theta\mathcal{J}}{N}}\right)^N = e^{\theta\mathcal{J}}$$

$$e^{\theta\mathcal{J}} = \sum_{n=0}^{\infty}\frac{\theta^n\mathcal{J}^n}{n!} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Two approaches to rotation

1, apply trigonometry, we can get $R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

2， what is left invariant?

$$\vec{u}' = R\vec{u}, \vec{v}' = R\,\vec{v} \quad \rightarrow \quad \vec{u}'^T \cdot \vec{v}' = \vec{u}^T \cdot \vec{v} \rightarrow R^T R = I$$

Lie: infinitesimal rotation -------- $e^{\theta \mathcal{J}} = R(\theta)$

Distance squared between neighboring points

- Check
- $\vec{r}_P = (x, y), \vec{r}_Q = (\tilde{x}, \tilde{y})$
- $\Delta x = \tilde{x} - x, \Delta y = \tilde{y} - y$
- $\begin{pmatrix} \Delta x' \\ \Delta y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$
- $\Delta x^2 + \Delta y^2 = \Delta x'^2 + \Delta y'^2 \rightarrow dx^2 + dy^2 = dx'^2 + dy'^2$

To higher-dimensional space

- $ds^2 = \sum_{i=1}^{N} (dx^i)^2$

$d\vec{x}' = R d\vec{x}$, requirement : ds^2 keep invariant.

$$R^T R = I$$

$$\det R = 1 : \quad eliminate\ reflection$$

- In $3d$, 3 generators:

- $\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$A = \theta_x \mathcal{J}_x + \theta_y \mathcal{J}_y + \theta_z \mathcal{J}_z$$

$$[J_i, J_j] = i\boldsymbol{\varepsilon_{ijk}} J_k$$

$$R(\theta) = e^A = e^{\sum_i \theta_i \mathcal{J}_i}$$

Structure constant

$$J = -i\mathcal{J} \rightarrow hermitian$$

$R(\theta) = e^A = e^{i\vec{\theta}\cdot\vec{J}}$: orthogonal matrix

Lie algebra

- $R \approx I + A,\ R' \approx I + B$: both infinitesimal rotation
- Consider $RR'R^{-1} - R'$: measure the lack of commutativity.
- The difference : $[A, B] \equiv AB - BA$
- $A = i \sum_i \theta_i J_i, \qquad B = i \sum_j \theta'_j J_j$

Then $\quad [A, B] = i^2 \sum_{ij} \theta_i\, \theta'_j [J_i, J_j],$

$[J_i, J_j]^T = -[J_i, J_j]$, antisymmetric, ok, as a linear combination of the $J_k S$

$$[J_i, J_j] = i\boldsymbol{\varepsilon_{ijk}} J_k$$

The commutation relations between the generators define a Lie algebra, with $\boldsymbol{\varepsilon_{ijk}}$ referred to as the structure constants of the algebra.

- Lie group: multiplication
- Lie algebra: commutation

As the generators： the antisymmetric matrices

$$C = AB, \quad C^T = B^T A^T = BA \neq -C$$

The algebra does not close under multiplication, only under commutation

- A lie algebra is a linear space spanned by linear combinations $\sum_i \theta_i \mathcal{J}_i$ of the generators.

Historically, the relation between a Lie group and Lie algebra was also hinted at by the Baker-Campbell-Hausdorff formula,

$$e^A e^B = e^C$$

$$C = A + B + \frac{1}{2}[A, B] + \frac{1}{12}\big([A, [A, B]] + [B, [B, A]]\big) + \cdots$$

In general higher-dimensional space

- $\left(J_{(mn)}\right)^{ij} = -i\left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right)$: generators $\left(J_{(mn)} = -J_{(nm)}\right)$
- $\mathcal{J}_{(mn)}$: $mth\ row, nth\ column$: 1 ; $nth\ row,\ mth\ column$: -1

  elsewhere: 0

- (mn) *label matrix* (denote which matrix) , *not matrix element :*
  i, j: for row and column $of\ \mathcal{J}_{(mn)}$

- $\frac{1}{2}N(N-1)$ real antisymmetric $N - by - N$ matrices $\mathcal{J}_{(mn)}$

  m: $N\ options,$ n: $(N-1)\ options,$ $\frac{1}{2}$: $double\ counting$

$$J_x = J_{23}, \quad J_y = J_{31}, \quad J_z = J_{12}$$

Try it! $\left(\mathcal{J}_{(mn)}\right)^{ij} = \left(\delta^{mi}\delta^{nj} - \delta^{mj}\delta^{ni}\right)$

$$\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & & \\ & & & \end{pmatrix} = \mathcal{J}_{(12)} \qquad \begin{pmatrix} & & & \\ & 0 & 1 & \\ & -1 & 0 & \\ & & & \end{pmatrix} = \mathcal{J}_{(23)} \qquad \begin{pmatrix} 0 & & -1 & \\ & & & \\ 1 & & 0 & \\ & & & \end{pmatrix} = \mathcal{J}_{(31)}$$

$$\begin{pmatrix} 0 & & & 1 \\ & & & \\ & & & \\ -1 & & & 0 \end{pmatrix} = \mathcal{J}_{(14)} \qquad \begin{pmatrix} & & & \\ & 0 & & 1 \\ & & & \\ & -1 & & 0 \end{pmatrix} = \mathcal{J}_{(24)} \qquad \begin{pmatrix} & & & \\ & & & \\ & & 0 & 1 \\ & & -1 & 0 \end{pmatrix} = \mathcal{J}_{(34)}$$

$The\ Lie\ algebra\ for\ SO(N)$

- $\left[J_{(mn)}, J_{(pq)}\right] = i\left(\delta_{mp}J_{(nq)} + \delta_{nq}J_{(mp)} - \delta_{np}J_{(mq)} - \delta_{mq}J_{(np)}\right)$

- (1) no integer in common, $= 0$
- (2) one integer in common, refer to $SO(3)$： 3-indices → 3-dim
- (3) two integers in common, $= 0$

The group $U(N) \to SU(N)$

- $U^\dagger U = 1, \to \det\left(U^\dagger U\right) = \det\left(U^\dagger\right)\det(U) = \det(U)^* \det(U) =$

$$|\det(U)|^2 = 1$$

- $\det(U) = e^{i\alpha}$

S: *special* $\to \det(U) = 1$: eliminates this dangling phase factor $e^{i\alpha}$

$$U(N): \begin{cases} e^{i\alpha} I \\ SU(N) \end{cases}$$

$$k = 1,2,\cdots,N$$

Strictly, $U(N) = (SU(N)/Z_N) \otimes U(1)$, $\because$ $\det\left(e^{\frac{i2\pi k}{N}} I\right) = 1$, but $e^{\frac{i2\pi k}{N}} I \subset U(1)$

U：幺正，幺模 → M：厄米矩阵，零迹矩阵

高阶小量

$$U \approx I + iM$$

$$(I + iM)^{\dagger}(I + iM) = I \Rightarrow I + i(M - M^{\dagger}) + M^{\dagger}M = I$$

$$M^{\dagger} = M$$

$$U = e^{iM}, \qquad \det U = 1$$

$$\det U = \det e^{iM} = \det e^{iW^{\dagger}\Lambda W} = \det W^{\dagger} e^{i\Lambda} W = \det e^{i\Lambda} = \prod_{j=1}^{N} e^{i\Lambda_j} = e^{i\,trM} = 1$$

M is traceless matrix

$SU(N)$:

$M^\dagger = M,$ $trM = 0$

独立参数约束条件：

非对角元，$(N^2 - N)/2 \ * 2$ Traceless： 1个约束

对角元，N （实数）

独立参数个数：$2N^2 - (N^2 - N + N + 1) = N^2 - 1$(减法)

反过来做加法也一样：

对角元 $N-1$（实，零迹）$+ (N^2 - N)/2$ (非对角元) $* 2$（实虚部）$= N^2 - 1$

$N = 2,$ 独立参数个数为3

The structure constants of the Lie algebra

- $U = e^{i\theta^a T^a}$

  a: $N^2 - 1$ values

$$[T^a, T^b]^\dagger = \left[{T^b}^\dagger, {T^a}^\dagger\right] = [T^b, T^a] = -[T^a, T^b] \text{(antihermitian)}$$

$$Tr\left([T^a, T^b]\right) = Tr\left(T^a T^b\right) - Tr\left(T^b T^a\right) = 0 \text{(traceless)}$$

$$\therefore\ [T^a, T^b] = i f^{abc} T^c$$

- $SU(2)$群元的一般形式：

$$u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = e^{iM}$$

- M幺正，零迹？　　　Complex matrix

$N = 2$

- Hermiticity: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = M$

  $b = c^*\ ,\ a\ and\ d\ real$

- Traceless: $a = -d$

$$\therefore M = \begin{bmatrix} a & b \\ b^* & -a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \theta_3 & \theta_1 - i\theta_2 \\ \theta_1 + i\theta_2 & -\theta_3 \end{bmatrix}$$

$$M = \frac{1}{2}\vec{\theta}\cdot\vec{\sigma} \quad , \ \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

Pauli matrices here!

$$S = \frac{\sigma}{2} \rightarrow [S_i, S_j] = i\epsilon_{ijk}S_k$$

Question:

- How to check the $SU(2)$ symmetry in a given Hamiltonian?

- How to construct a MPS with $SU(2)$ symmetry?

- M由零迹的厄米矩阵生成，我们将 $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ 称为$SU(2)$ 群的无穷小生成元。

- $SU(2)$ 群中任一个与单位元无限接近的元素，可用三个无穷小实参数 $\delta_x, \delta_y, \delta_z$ 匹配生成元生成，即为 $I - i\vec{\delta} \cdot \vec{\sigma}$

对易关系： $[\sigma_x, \sigma_y] = 2i\sigma_z, [\sigma_y, \sigma_z] = 2i\sigma_x, [\sigma_z, \sigma_x] = 2i\sigma_y$

$$S = \frac{\sigma}{2} \rightarrow [S_i, S_j] = i\epsilon_{ijk}S_k$$

- 无穷小生成元 $\sigma_x, \sigma_y, \sigma_z$ 连同它们满足的对易关系，构成李代数$su(2)$。

- $su(2)$的任一群元X可以写为

$$X = x\sigma_x + y\sigma_y + z\sigma_z = \vec{r} \cdot \vec{\sigma}, \qquad x, y, z \in R$$

$SU(2)$群在单位元 I 邻域的元素$I - i\vec{\delta} \cdot \vec{\sigma}$

与该群李代数的元素 $\vec{\delta} \cdot \vec{\sigma}$

有一一对应关系: isomorphic

$N = 3$

Traceless, unitary matrices

- $\theta^{\alpha}\ \lambda^{\alpha} (\alpha = 1,2, \ldots, 8)$

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$ ---- Gell-mann matrices

We have tr $(\sigma_a \sigma_b) = 2\delta_{ab}$ and tr $(\lambda_a \lambda_b) = 2\delta_{ab}$, then $U = e^{i\theta^a \lambda^a / 2}$

$SO(3)$群： 行列式为1的三阶实正交矩阵O，满足

$$O^tO = I, \det O = 1$$

- 与上述类似，考虑与单位元I无限接近的$SO(3)$群元。

设M 是无穷小的三阶实矩阵，即M的每个矩阵元都是无穷小的实数，则与E无穷接近的$SO(3)$群元记作$I + M$。由正交条件得

$$M^t = -M$$

三维实空间独立的反对称矩阵可取为

$$\mathcal{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \mathcal{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathcal{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- 改成厄米矩阵 $\vec{J}=(J_x,J_y,J_z)$

$$J_x=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix},J_y=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix},J_z=\begin{pmatrix}0&i&0\\-i&0&0\\0&0&0\end{pmatrix}$$

$SO(3)$群中与单位元无限接近的元素可写为$I-i\delta\vec{\psi}\cdot\vec{J}$

与其李代数的元素$\delta\vec{\psi}\cdot\vec{J}$一一对应

$SU(2)$ is locally isomorphic to $SO(3)$

- 共同点：三个独立参数

（1） 矩阵h

（a） 二维迹为0的厄米矩阵，是 pauli 矩阵的线性组合

已知 pauli 矩阵

$$\sigma_x=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\sigma_y=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\sigma_z=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$h=\begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}=x\sigma_x+y\sigma_y+z\sigma_z=\vec{r}\cdot\vec{\sigma}$$

$$=\begin{bmatrix}z & x-iy\\ x+iy & -z\end{bmatrix}$$

$$h=h^{\dagger}$$

$$x=\frac{h_{12}+h_{21}}{2}, y=\frac{h_{21}-h_{12}}{2i}, z=h_{11}=-h_{22}$$

(b) 用 u 矩阵对 h 做幺正变换

$$h'=uhu^{-1}=\begin{bmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{bmatrix}=\vec{r}'\cdot\vec{\sigma}$$

$$u=\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} \Rightarrow u^{-1}=\begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix}$$

$$h'=uhu^{-1}=\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}\begin{bmatrix} z & x-iy \\ x+iy & -z \end{bmatrix}\begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix}$$

$$h = \vec{r} \cdot \vec{\sigma}$$

$$h' = uhu^{-1}$$

$$h' = \vec{r}' \cdot \vec{\sigma}$$

$$\vec{r} \rightarrow \vec{r}' \qquad \vec{r}' = R(u)\vec{r}$$

每一个使矩阵 h 变成 h' 的幺模幺正阵 u ， 总存在一个 3×3

的矩阵$R(u)$ 使得$\vec{r}(x,y,z)$ 变成 $\vec{r}'(x',y',z')$

$$R(u) = \begin{pmatrix} \frac{1}{2}(a^2 + {a^*}^2 - b^2 - {b^*}^2) & \frac{-i}{2}(a^2 - {a^*}^2 + b^2 - {b^*}^2) & -(a^*b^* + ab) \\ \frac{i}{2}(a^2 - {a^*}^2 - b^2 + {b^*}^2) & \frac{1}{2}(a^2 + {a^*}^2 + b^2 + {b^*}^2) & i(a^*b^* - ab) \\ a^*b + ab^* & i(a^*b - ab^*) & aa^* - bb^* \end{pmatrix}$$

举例

- 例一

u是对角阵

$$u=\begin{bmatrix} a & 0 \\ 0 & a^* \end{bmatrix} \qquad aa^*=1$$

取$a=e^{-i\frac{\alpha}{2}}$

$$u_1(\alpha)=\begin{bmatrix} e^{-i\frac{\alpha}{2}} & 0 \\ 0 & e^{i\frac{\alpha}{2}} \end{bmatrix}$$

$$R_1(\alpha)=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow R(z,\alpha)$$

- 例二　　选择一个实矩阵

$$u_2(\beta) = \begin{bmatrix} \cos\frac{\beta}{2} & -\sin\frac{\beta}{2} \\ \sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{bmatrix}$$

$$R_2(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \quad \to R(y,\beta)$$

- （2）　$R(u)$ 的性质

（a）$R(u)$是个实矩阵：r_i', r_i 都是实数

$$r_i' = \sum_j R(u)_{ij}\, r_j$$

（b）$R(u)$是转动矩阵：

幺正变换不改变矩阵的行列式

$$\det h' = \det u \cdot \det h \cdot \det u^{-1} = \det h$$

$$\det h = -z^2 - (x+iy)(x-iy) = -(x^2+y^2+z^2)$$

$$\det h' = -(x'^2+y'^2+z'^2)$$

$$x'^2+y'^2+z'^2 = x^2+y^2+z^2$$

- （c） $\det R\,(u) = 1$

转动矩阵的行列式为1或-1，这里所有的转动矩阵，可由矩阵 u 的参量 (a,b) 连续变化而得，

$$(1,0) \to u = I_0 \to \det R\,(I_0) = 1$$

行列式不能发生由1到-1的跳变，对于所有参量(a,b)都有

$$\det R\,(u) = 1$$

$R(u)$ ： 三维空间的正当转动

（3）$SU(2)$ 与 $SO(3)$ 同态

已知欧拉角表征的 任一 正当转动：

$$R(\alpha,\beta,\gamma) = R(z,\alpha)R(y,\beta)R(z,\gamma)$$

$$u(\alpha,\beta,\gamma) = u_1(\alpha)\ u_2(\beta)\ u_1(\gamma)$$

$$= \begin{bmatrix} e^{-i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} & -e^{-i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} \\ e^{i\frac{\alpha-\gamma}{2}}\sin\frac{\beta}{2} & e^{i\frac{\alpha+\gamma}{2}}\cos\frac{\beta}{2} \end{bmatrix}$$

$SU(2)$ covers $SO(3)$ twice

- 零迹厄米矩阵 h 作为媒介

$$h' = u\, h u^{-1}$$

$$\downarrow$$

$$\vec{r}' = R(u)\, \vec{r}$$

- but

$$h' = (-u)\, h(-u^{-1})$$

$$\downarrow$$

$$\vec{r}' = R(u)\, \vec{r}$$

$$\left.\begin{array}{l} u \\ -u \end{array}\right\} \rightarrow R(u)$$

2:1 的同态关系

Identity

- $$\exp\left(\frac{i\vec{\sigma}\cdot\vec{n}\alpha}{2}\right) = \left[1 - \frac{(\vec{\sigma}\cdot\vec{n})^2}{2!}\left(\frac{\alpha}{2}\right)^2 + \frac{(\vec{\sigma}\cdot\vec{n})^2}{4!}\left(\frac{\alpha}{2}\right)^4 - \cdots\right] + i\left[(\vec{\sigma}\cdot\vec{n})\frac{\alpha}{2} - \frac{(\vec{\sigma}\cdot\vec{n})^3}{3!}\left(\frac{\alpha}{2}\right)^3 + \cdots\right]$$
$$= \mathbf{1}\cos\left(\frac{\alpha}{2}\right) + \boldsymbol{i}\,(\vec{\sigma}\cdot\vec{n})\sin\left(\frac{\alpha}{2}\right)$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$(\vec{\sigma}\cdot\vec{\hat{n}})^n = \begin{cases} 1 & for\ n\ even \\ \vec{\sigma}\cdot\vec{\hat{n}} & for\ n\ odd \end{cases}$$

You can check

$U = \exp\left(\frac{i\vec{\sigma}\cdot\vec{n}\alpha}{2}\right)$

$U(2\pi) = -I, \; U(4\pi) = I$ *: double covering, only locally isomorphic*

- $U^\dagger \sigma_1 U = cos\alpha\sigma_1 + sin\alpha\sigma_2$
- $U^\dagger \sigma_2 U = -sin\alpha\sigma_1 + cos\alpha\sigma_2$
- $U^\dagger \sigma_3 U = \sigma_3$

Half angles come in to render the Lie algebra of $SU(2)$ and $SO(3)$
the same !

SU(2) 转动→ $SO(3)$转动

Calculate :

$$U^{\dagger}\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{\sigma}}U$$
$$= x(cos\alpha\sigma_1 + sin\alpha\sigma_2) + \mathrm{y}(-sin\alpha\sigma_1 + cos\alpha\sigma_2) + \mathrm{z}\sigma_3$$
$$= (\mathrm{x}cos\alpha - \mathrm{y}sin\alpha)\sigma_1 + (xsin\alpha + ycos\alpha)\sigma_2 + \mathrm{z}\sigma_3 = \vec{\boldsymbol{r}}' \cdot \vec{\boldsymbol{\sigma}}$$

$$\vec{\boldsymbol{r}}' = \boldsymbol{R}(\boldsymbol{z},\boldsymbol{\alpha})\vec{\boldsymbol{r}}$$

$SO(3)$转动→ SU(2)转动

由自旋pauli矩阵张开的代数空间（2维厄米零迹）， 测量算符由$\widehat{\boldsymbol{O}} = \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{\sigma}} = x\hat{\boldsymbol{\sigma}}_x + y\hat{\boldsymbol{\sigma}}_y + z\hat{\boldsymbol{\sigma}}_z$表示。设初始$\vec{\boldsymbol{r}} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$，$\vec{\boldsymbol{r}}$ 经过旋转变换（绕着$\boldsymbol{y}$轴转$\boldsymbol{\theta}$角）得到$\vec{\boldsymbol{r}}'$，求算符$\widehat{\boldsymbol{O}}' = \vec{\boldsymbol{r}}' \cdot \vec{\boldsymbol{\sigma}}$的本征矢量。

$$\widehat{\boldsymbol{O}}' = \vec{r}' \cdot \vec{\sigma} = \sin\theta\,\hat{\sigma}_x + \cos\theta\,\hat{\sigma}_z$$

对应关系

$$\vec{r} = (0,0,1) \Rightarrow \vec{r}' = (\sin\theta\,, 0, \cos\theta)$$

$$|\uparrow\rangle \Rightarrow |+\rangle = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle$$

$\hat{\boldsymbol{O}}'|+\rangle = |+\rangle$

- 三维空间中 $\vec{r}$ 矢量的θ角旋转，
- 自旋空间中本征矢$|\uparrow\rangle$ 的 $\theta/2$角旋转。

- Group is different, algebra is the same
- Locally isomorphic

$$N^2 - 1 = \frac{M(M-1)}{2} \to$$

- $SU(2) \quad Vs \quad SO(3)$

$$N = 2, M = 3$$

- $SU(4) \quad Vs \quad SO(6)$

$$N = 4, M = 6$$

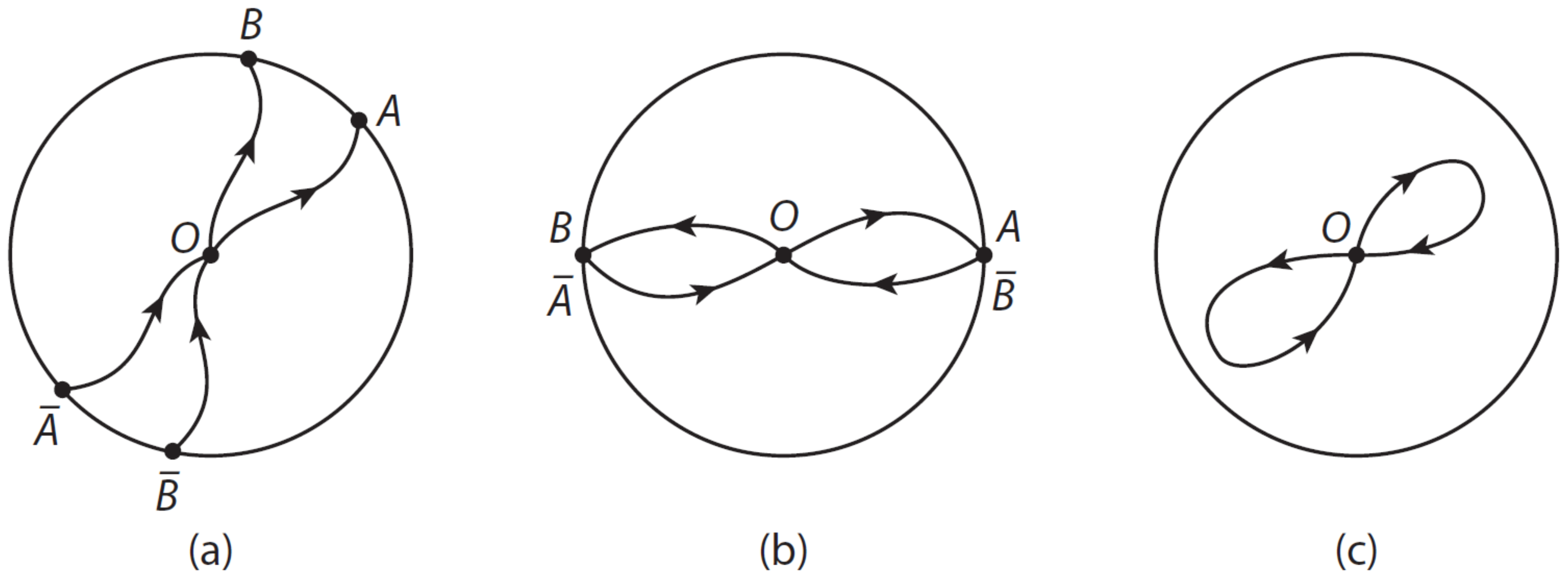
Manifold of Lie group $\pi_1\ (M)$

The (first) homotopy (同伦) group $\pi_1\ (M)$ of a manifold M

Two closed curves in a manifold are said to be homotopic to each other if they can be continuously deformed into each other.

$SO(3)$

- $\pi_1(SO(3)) = Z_2$ (双连通)



$$R(\vec{\omega}, \alpha) = R(-\vec{\omega}, \ 2\pi - \alpha)$$

$\alpha = \pi$: the same element→one curve one jump($O \to A, \bar{A} \to O$)

With even jumps , can continuous deform to one point

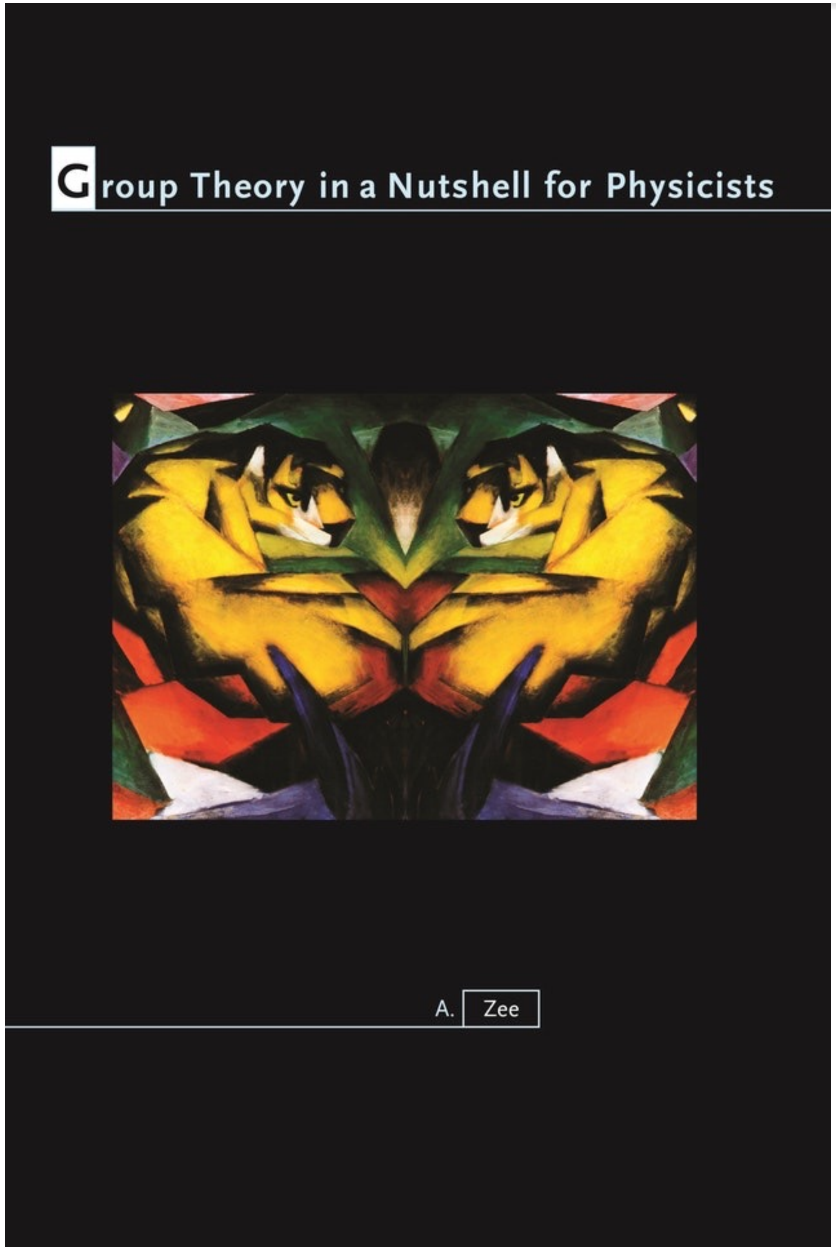
odd jumps, can not continuous deform to one point

- $\pi_1\left(SU(2)\right) = \emptyset$ (can be shrunk to one point)

  $u(\vec{\omega}, \alpha) = -u(-\vec{\omega}, 2\pi - \alpha)$

  $\alpha = \pi$: different elements : the two points connected by the origin in the sphere

- This topological consideration confirms what we already know, that $SU(2)$ is locally isomorphic to $SO(3)$, but since $SU(2)$ double covers $SO(3)$, they cannot possibly be globally isomorphic.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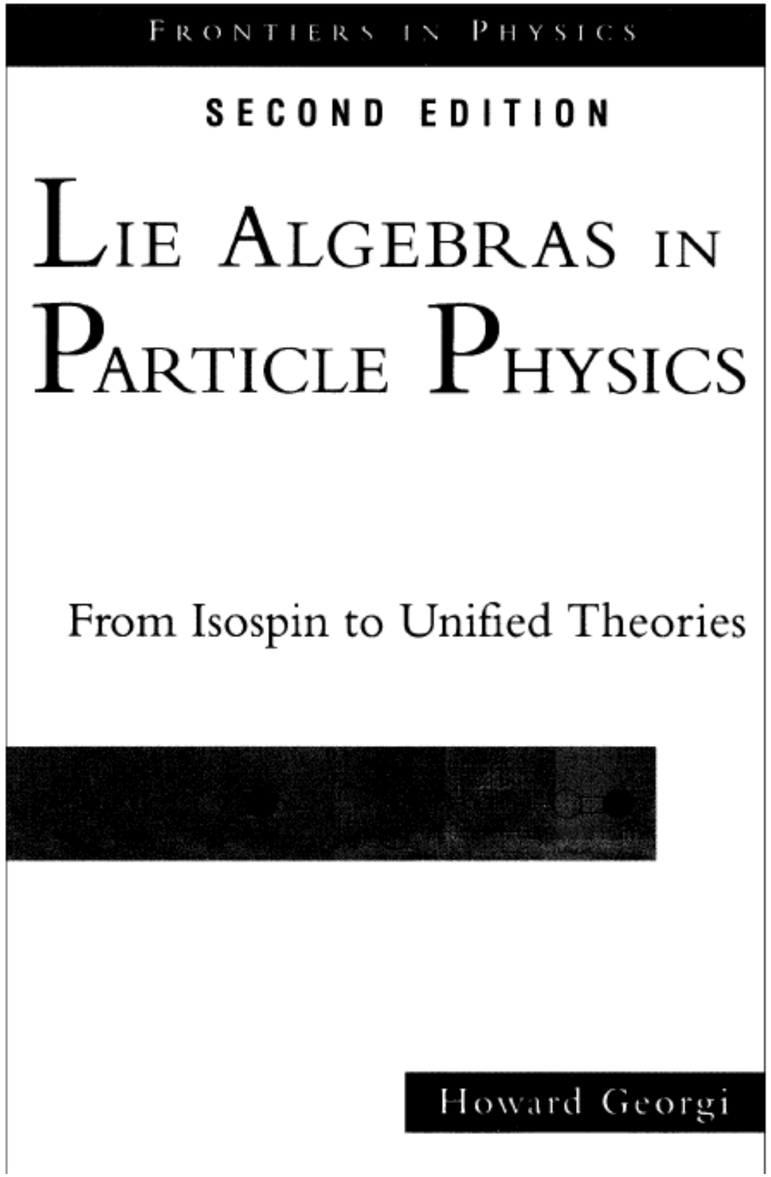
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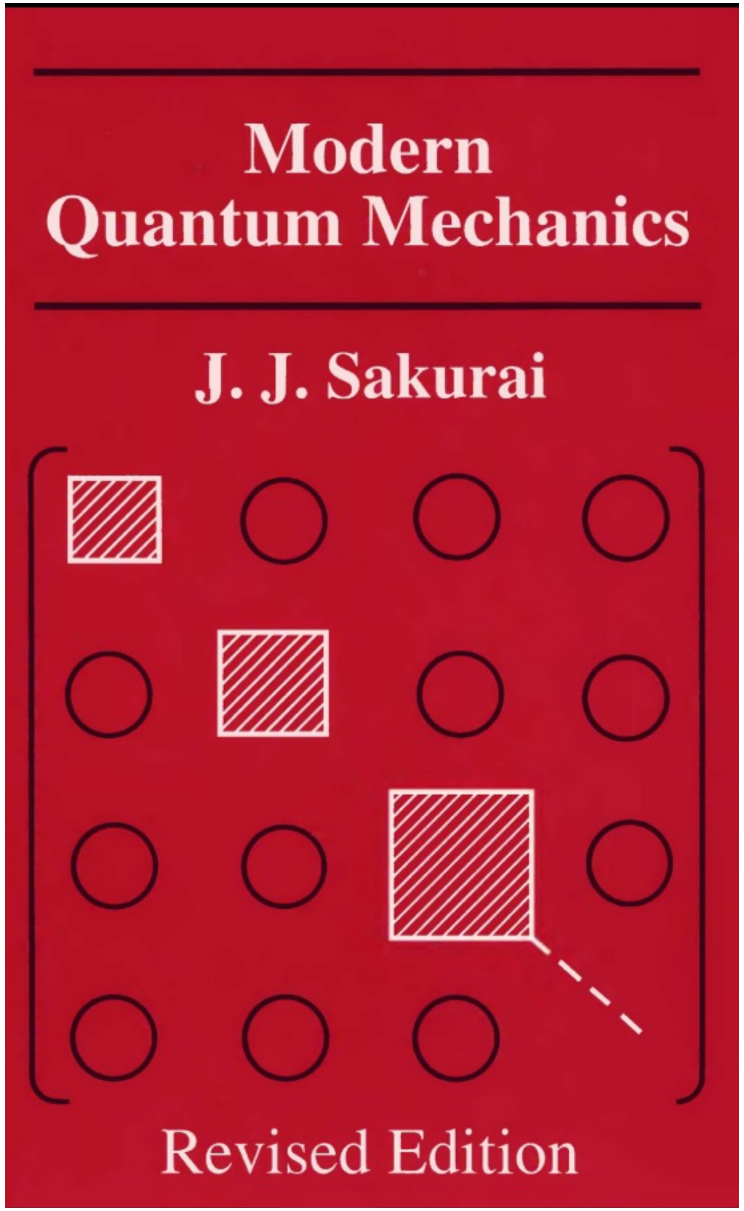
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