徐婉棠书中关于δ函数处理有误，现在考写其过程，整理如下：

已知表示矩阵元的完备性关系为

$$\sum_{i\alpha\gamma}\frac{l_i}{g}D^{i}(S)^{*}_{\alpha\gamma}\,D^{i}(T)_{\alpha\gamma}=\delta_{ST}$$

证：设 S，T 分别为两个类 C_l，和 C_m 中的类元

并对两个类中类元全部求和，得

$$\sum_{i\alpha\gamma}\frac{l_i}{g}\sum_{S\in C_l}D^{i*}(S)_{\alpha\gamma}\sum_{T\in C_m}D^{i}_{\alpha\gamma}(T)$$

$$=\sum_{S\in C_l}\sum_{T\in C_m}\delta_{ST} \qquad (*)$$

RHS：$l\neq m$ 时为 0

当 $l=m$，不为 0

$$=\sum_{S\in C_l}\delta_{SS}=h_l$$

$\therefore$ RHS $=h_l\,\delta_{lm}$

令 $\sum_{s\in C_l} D^i(s) = M^{(i,l)}$

那么 $[M^{(i,l)}, D^i(R)] = 0$

due to $D^i(R^{-1})[M^{(i,l)}, D^i(R)] = 0$

M 是整个类求和得到的矩阵，

根据 Schur 引理

$$M^{(i,l)} = C^{(i,l)} I_0$$
$$= \sum_{s\in C_l} D^i(s)$$

两边取迹

$$tr(M^{(i,l)}) = \sum_{s\in C_l} tr(D^i(s))$$
$$= \sum_{s\in C_l} \chi^i(s) = h_l \chi^i(C_l)$$

$tr(C^{(i,l)} I_0) = C^{(i,l)} l_i$ （第 i 个不可约表示的维度）

$\therefore$ 常数 $C^{(i,l)} = \frac{h_l}{l_i} \chi^i(C_l)$

代回 $m^{(i,l)}$ 表达式

$$m^{(i,l)} = \frac{h_l}{l_i}\chi^i(C_l)\, I_0$$

则 $$m^{(i,l)}_{\alpha\gamma} = \frac{h_l}{l_i}\chi^i(C_l)\,\delta_{\alpha\gamma}$$

代入(★)左边

$$\sum_{\underbrace{i\alpha\gamma}_{\sum_{i=1}^{r}\sum_{\alpha=1}^{l_i}\sum_{\gamma=1}^{l_i}}} \frac{l_i}{g}\,\frac{h_l}{l_i}\chi^{i*}(C_l)\,\frac{h_m}{l_i}\chi^i(C_m)\,\delta_{\alpha\gamma}\delta_{\alpha\gamma}$$

$$= \sum_i\sum_\alpha \frac{h_l h_m}{g\, l_i}\chi^{i*}(C_l)\,\chi^i(C_m)\,\delta_{\alpha\alpha}$$

$$= \sum_i \frac{h_l h_m}{g}\chi^{i*}(C_l)\,\chi^i(C_m)$$

(★)右边 $= h_l\delta_{lm} = h_m\delta_{lm}$

$$\therefore \quad \sum_i h_l\, \chi^{i*}(C_l)\,\chi^i(C_m) = g\,\delta_{lm}$$

得证 特征标的完全性关系。

这一关系在 Howard George 的书里有一个巧妙的处理。

that $F(g_1)$ can be expanded in terms of the matrix elements of the irreducible representations —

$$F(g_1) = \sum_{a,j,k} c^a_{jk} [D_a(g_1)]_{jk} \tag{1.81}$$

but since F is constant on conjugacy classes, we can write it as

$$F(g_1) = \frac{1}{N} \sum_{g\in G} F(g^{-1} g_1 g) = \frac{1}{N} \sum_{a,j,k} c^a_{jk} [D_a(g^{-1} g_1 g)]_{jk} \tag{1.82}$$

and thus

$$F(g_1) = \frac{1}{N} \sum_{\substack{a,j,k \\ g,\ell,m}} c^a_{jk} [D_a(g^{-1})]_{j\ell} [D_a(g_1)]_{\ell m} [D_a(g)]_{mk} \tag{1.83}$$

But now we can do the sum over g explicitly using the orthogonality relation, (1.68).

$$F(g_1) = \sum_{\substack{a,j,k \\ \ell,m}} \frac{1}{n_a} c^a_{jk} [D_a(g_1)]_{\ell m} \delta_{jk} \delta_{\ell m} \tag{1.84}$$

or

$$F(g_1) = \sum_{a,j,\ell} \frac{1}{n_a} c^a_{jj} [D_a(g_1)]_{\ell\ell} = \sum_{a,j} \frac{1}{n_a} c^a_{jj} \chi_a(g_1) \tag{1.85}$$

This was straightforward to get from the orthogonality relation, but it has an important consequence. The characters, $\chi_a(g)$, of the independent irreducible representations form a complete, orthonormal basis set for the functions that are constant on conjugacy classes. Thus the number of irreducible representations is equal to the number of conjugacy classes. We will use this frequently.

This also implies that there is an orthogonality condition for a sum over representations. To see this, label the conjugacy classes by an integer α, and let k_α be the number of elements in the conjugacy class. Then define the matrix V with matrix elements

α : label class

$$V_{\alpha a} = \sqrt{\frac{k_\alpha}{N}} \chi_{D_a}(g_\alpha) \tag{1.86}$$

a : label irre reps

where g_α is the conjugacy class α. Then the orthogonality relation (1.79) can be written as $V^\dagger V = 1$. But V is a square matrix, so it is unitary, and thus we also have $VV^\dagger = 1$, or

$$\sum_a \chi_{D_a}(g_\alpha)^* \chi_{D_a}(g_\beta) = \frac{N}{k_\alpha} \delta_{\alpha\beta} \tag{1.87}$$

→ 特征标表中的行/列向量