

南方科技大学

物理系

Department of Physics
SUSTech

Time reversal symmetry

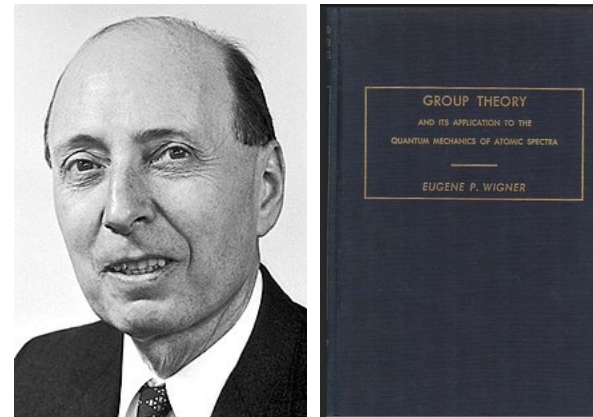
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Time reversal Θ

Definition:

Eugene Wigner (1902 – 1995), 《Group Theory and Its Applications to Quantum Mechanics of Atomic Spectra》, Academic Press (1959).

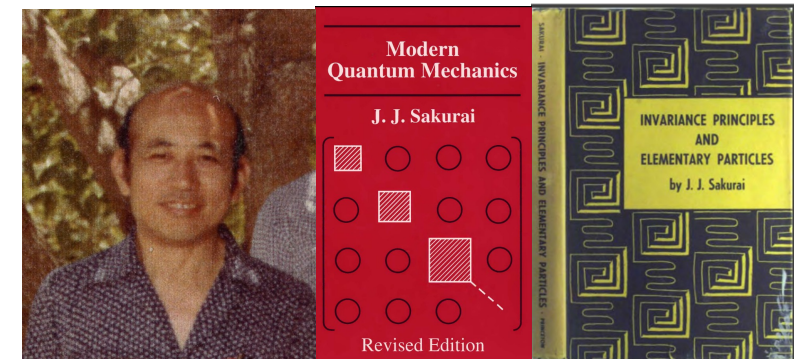


Most inspiring demonstration:

Jun John Sakurai (櫻井純, 1933-1982)

《Modern quantum mechanics》

《Invariance principle and elementary particles》



Newtonian mechanics

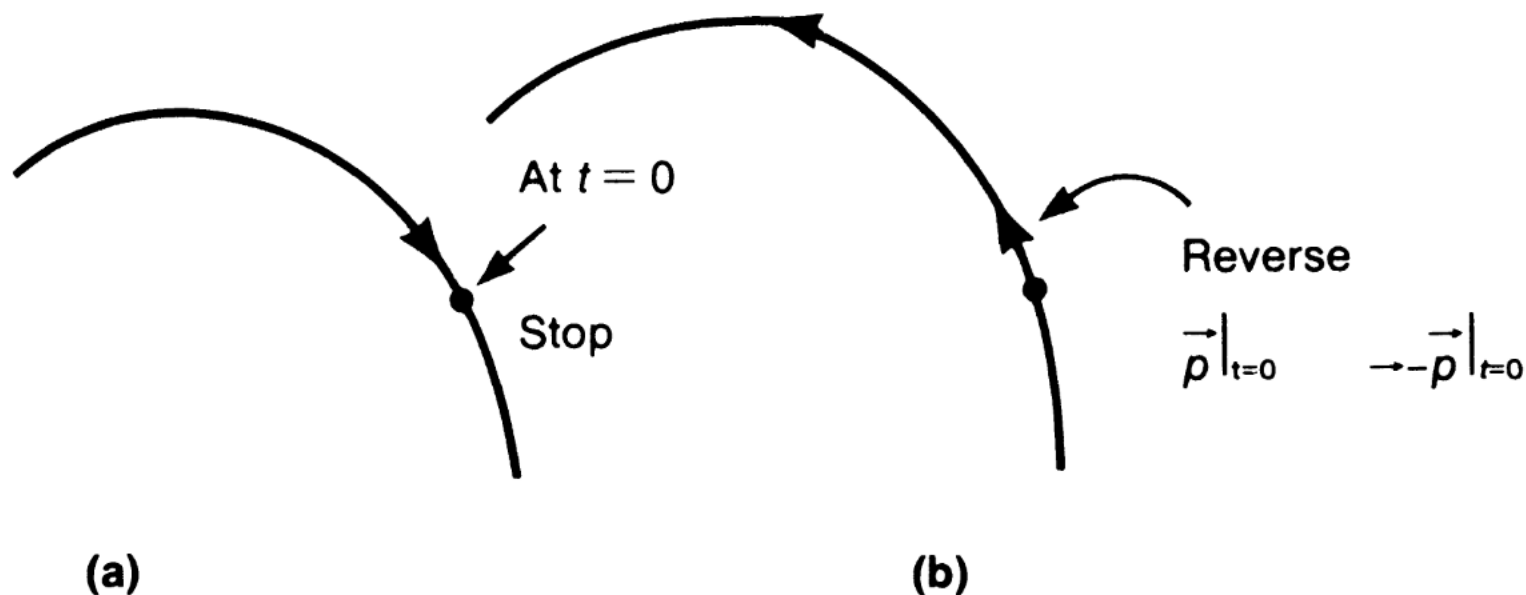


FIGURE 4.9. (a) Classical trajectory which stops at $t = 0$ and (b) reverses its motion $\mathbf{p}|_{t=0} \rightarrow -\mathbf{p}|_{t=0}$.

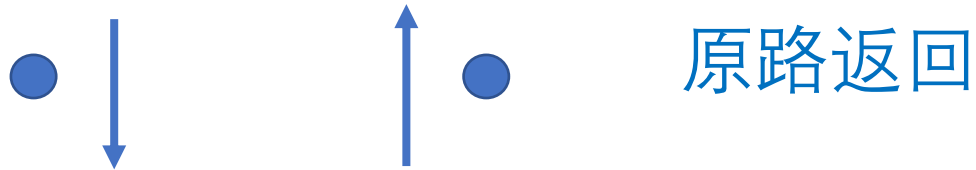
$t = 0$ 处：开始运动的反演，速度反向

Newtonian mechanics

$$m \frac{d^2 \vec{x}}{dt^2} = \vec{F}(\vec{x})$$

Clearly if $\vec{x}(t)$ is a possible trajectory, then likewise so is $\vec{x}(-t)$ since

$$m \frac{d^2 \vec{x}}{d(-t)^2} = \vec{F}(\vec{x})$$



Falling body vs thrown upwards

问：量子力学的“原路返回”是什么呢？

Wigner:

1932: 提出“时间反演”的概念

1959: “reversal of the direction of motion” is perhaps a more felicitous, though longer, expression than “time inversion.”

Sakurai:

1994 (遗作) : time reversal is a misnomer, it reminds us of science fiction.

Wigner(1932,1959): out of dilemma

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \hat{H} \psi(\vec{r}, t)$$

$$\psi(\vec{r}, t) = e^{i(\vec{p} \cdot \vec{x} - E_n t)/\hbar} \quad t \rightarrow -t \quad \psi(\vec{r}, t) = e^{i(\vec{p} \cdot \vec{x} + E_n t)/\hbar}$$

• Contradiction: $\frac{p^2}{2m} \psi(\vec{r}, t) = E_n \psi(\vec{r}, t) \quad \frac{p^2}{2m} \psi(\vec{r}, t) = -E_n \psi(\vec{r}, t)$

Introduce time reversed state $\Theta \psi(\vec{r}, t) \stackrel{\text{def}}{=}$

$$e^{-i(\vec{p} \cdot \vec{x} - E_n(-t))/\hbar} = e^{i(-\vec{p} \cdot \vec{x} - E_n t)/\hbar}$$

$$\Theta |p\rangle = |-p\rangle, E_n \text{ keep invariant!}$$

- We require the probability of finding a particle which will be the same .

$$\langle \psi | \psi \rangle = \langle \Theta \psi | \Theta \psi \rangle$$

We have two choices : $|\langle \varphi | \psi \rangle| = |\langle \Theta \varphi | \Theta \psi \rangle|$

$$\langle \varphi | \psi \rangle = \langle \Theta \varphi | \Theta \psi \rangle \quad (\Theta \text{ unitary, keep inner product invariant})$$

$$\langle \varphi | \psi \rangle^* = \langle \psi | \varphi \rangle = \langle \Theta \varphi | \Theta \psi \rangle \quad (\Theta \text{ anti-unitary: an extra complex conjugate})$$

Hermitian operator

→ a generalization of a real number

Hermitian conjugate of $|\psi\rangle$

$$|\psi\rangle^\dagger = \langle\psi|$$

$$\langle\psi|^\dagger = |\psi\rangle$$

Why this requirement $\hat{O} = \hat{O}^\dagger$ (Hermitian, or self-adjoint) ?

$$a = \langle\psi|\hat{O}|\psi\rangle = \langle\psi|\hat{O}^\dagger|\psi\rangle = a^* \quad (\text{physical observable})$$

$$\text{Generally, } \langle\varphi|\hat{O}|\psi\rangle = \langle\psi|\hat{O}|\varphi\rangle^*$$

反线性么正算符 Θ

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \hat{H} \psi(\vec{r}, t)$$

数学处理: $t \rightarrow -t$

$$i\hbar \frac{\partial \psi(\vec{r}, -t)}{\partial(-t)} = H \psi(\vec{r}, -t), \text{ 不满足薛定谔方程}$$

等式两边取复共轭, 发现

当 $\psi'(\vec{r}, t) = \psi^*(\vec{r}, -t)$ 满足薛定谔方程,

回到对于时间反演算符 Θ (物理)的要求, 将时间反演算符作用于方程的两边

$$\Theta i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial(t)} = \Theta H \psi(\vec{r}, t)$$

如果 $\Theta = UK$ 是反线性的么正算符, 问题迎刃而解, Θ 与 i 的对易关系负号合并数学上 $t \rightarrow -t$ 的要求。

$$\text{左边} = -i\hbar \Theta \frac{\partial \psi(\vec{r}, t)}{\partial(t)} = \text{右边} = H \Theta \psi(\vec{r}, t)$$

$$\left[\Theta, i \frac{\partial}{\partial t} \right] = 0 \text{ match } [\Theta, H] = 0$$

如果 $\Theta \psi(\vec{r}, t) = \psi^*(\vec{r}, -t)$

一切变得简单，复共轭视为打补丁（秦思学）

得到 $\psi^*(\vec{r}, -t)$ 满足薛定谔方程的形式。

复共轭不改变 跃迁概率 (transition probability)

$$\left(1 - \frac{iH}{\hbar} \delta t\right) \Theta |\alpha\rangle = \Theta \left(1 - \frac{iH}{\hbar} (-\delta t)\right) |\alpha\rangle$$

$$-iH\Theta = \Theta iH$$

If Θ is unitary,

$$-H\Theta = \Theta H$$

$$H\Theta |n\rangle = -\Theta H |n\rangle = -E_n \Theta |n\rangle$$

Nonsensical!



Θ is anti-unitary \rightarrow antilinear

$$-iH\Theta = \Theta iH = -i\Theta H$$

$$H\Theta = \Theta H$$

时间反演算符的性质

- 反线性

$$\Theta(a\phi + b\varphi) = a^* \Theta\phi + b^* \Theta\varphi$$

$$(\Theta\phi, \Theta\varphi) = (\varphi, \phi) = (\phi, \varphi)^*$$

- 反么正（反线性的么正）

$$\Theta = UK$$

Unitary * complex conjugate

再问：量子力学的“原路返回”是什么呢？

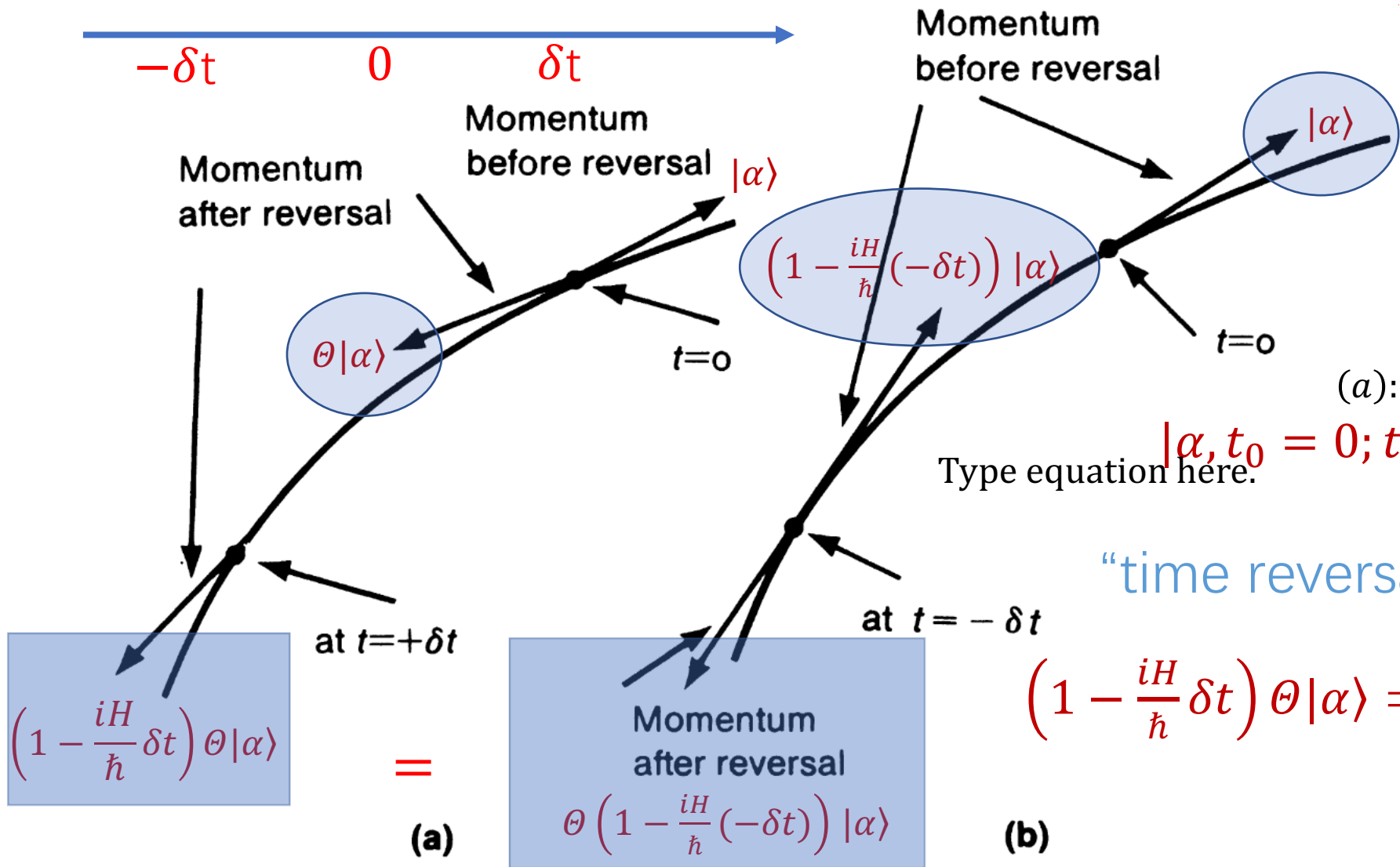


FIGURE 4.11. Momentum before and after time reversal at time $t = 0$ and $t = \pm \delta t$.

Note:

All the treatments start at $t = 0$

(a): $0 \rightarrow \delta t$

Type equation here. $|\alpha, t_0 = 0; t = \delta t\rangle = (1 - \frac{iH}{\hbar} \delta t) |\alpha\rangle$

“time reversal symmetry” requires

$$\begin{aligned}
 (1 - \frac{iH}{\hbar} \delta t) \Theta |\alpha\rangle &= \Theta |\alpha, t_0 = 0; t = -\delta t\rangle \\
 &= \Theta (1 - \frac{iH}{\hbar} (-\delta t)) |\alpha\rangle
 \end{aligned}$$

(b): $(-\delta t \rightarrow 0)^{-1}$

“reversal of motion”

希尔伯特空间 state的“原路返回”

时间倒流了吗？

你不在的时候，我有个机会去过了一段年轻时候的日子。本来以为我再活一次的话，也许会有什么不一样，结果，还是差不多，没什么不同。只是突然觉得，再活一次的话，好像真的没那个必要，真的没那个必要。



你知道我以后想做什么吗？我要去告诉别人他们不知道的事，给别人看他们看不到的东西，我想，这样一定天天都很好玩。



《 A one and a Two 》：杨德昌 的 《一一》

所谓“倒带”

- 不是 $0 \rightarrow t$ vs $t \rightarrow 0$
- 而是 $-t \rightarrow 0$ vs $0 \rightarrow t$

时间是一直流逝着的。

态是时间的函数，但时间不是态的属性。

Goal of function

- Under the transformation of time reversal: $t \rightarrow -t$

$$x \rightarrow x, p = \frac{dx}{dt} \rightarrow -p,$$

$$L = r \times p \rightarrow -L, s \rightarrow -s \implies J = L + s \rightarrow -J$$

$$[x, p] = i\hbar$$

Odd/even operator

$$\Theta A \Theta^{-1} = \pm A$$

The operation of time-reversal operator

Dual
Corres-
pondence

proof:

$$\langle \beta | \Theta | \alpha \rangle = \langle \beta | (\Theta | \alpha \rangle)$$

$$|\tilde{\alpha}\rangle = \Theta |\alpha\rangle, \quad |\tilde{\beta}\rangle = \Theta |\beta\rangle,$$

$$|\gamma\rangle \equiv \otimes^\dagger |\beta\rangle$$

$$|\gamma\rangle \leftrightarrow \langle \gamma| = \langle \beta| \otimes$$

$$\langle \beta| \otimes |\alpha\rangle = \langle \gamma|\alpha\rangle = \langle \tilde{\alpha}|\tilde{\gamma}\rangle$$

$$= \langle \tilde{\alpha} | \Theta \otimes^\dagger | \beta \rangle$$

$$= \langle \tilde{\alpha} | \Theta \otimes^\dagger \Theta^{-1} \Theta | \beta \rangle$$

$$= \langle \tilde{\alpha} | \Theta \otimes^\dagger \Theta^{-1} | \tilde{\beta} \rangle$$

← Identity from the antiunitary nature of Θ

$$\langle \beta | \otimes |\alpha\rangle = \langle \tilde{\alpha} | \Theta \otimes^\dagger \Theta^{-1} | \tilde{\beta} \rangle$$



Linear operator

For Hermitian observables:

$$\langle \beta | A | \alpha \rangle = \langle \tilde{\alpha} | \Theta A \Theta^{-1} | \tilde{\beta} \rangle$$

Odd/even operator

$$\Theta A \Theta^{-1} = \pm A$$

$$\langle \beta | A | \alpha \rangle = \pm \langle \tilde{\beta} | A | \tilde{\alpha} \rangle^*$$

Expectation value

$$\langle \alpha | p | \alpha \rangle = -\langle \tilde{\alpha} | p | \tilde{\alpha} \rangle$$

$$\Theta p \Theta^{-1} = -p$$

$$p \Theta |p'\rangle = -\Theta p |p'\rangle = -\Theta p' |p'\rangle = -p' \Theta |p'\rangle$$

$\Theta |p'\rangle = | -p' \rangle$ (up to a phase factor): $| -p' \rangle$ is the momentum eigen-ket with eigenvalue $-p'$

$$\langle \alpha | x | \alpha \rangle = \langle \tilde{\alpha} | x | \tilde{\alpha} \rangle$$

$$\Theta x \Theta^{-1} = x$$

$$\Theta |x'\rangle = |x'\rangle \text{ (up to a phase)}$$

- Check the fundamental **commutation relation**:

$$[x_i, p_j] | \rangle = i\hbar \delta_{ij} | \rangle$$

$$\Theta [x_i, p_j] \Theta^{-1} \Theta | \rangle = \Theta i\hbar \delta_{ij} | \rangle$$

$$[x_i, -p_j] \Theta | \rangle = -i\hbar \delta_{ij} \Theta | \rangle$$

for angular-momentum operator,

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$$

$$\Theta J \Theta^{-1} = -J$$

- This is consistent for spin-less system where $J = r \times p$

Wave function

$|\alpha\rangle$: a spinless single – particle state

$$|\alpha\rangle = \int d^3x' |x'\rangle\langle x'|\alpha\rangle$$
$$\Theta|\alpha\rangle = \int d^3x' |x'\rangle\langle x'|\alpha\rangle^*$$
$$\psi(x') \rightarrow \psi^*(x')$$

Theorem : Suppose the Hamiltonian is invariant under time reversal and the energy eigen-ket $|n\rangle$ is nondegenerate; then the corresponding Energy eigen-function is real (or ,more generally, a real function times a phase factor independent of \mathbf{x})

$$\langle x'|n\rangle = \langle x'|n\rangle^*$$

The eigen-**spinor** of $\sigma \cdot \hat{n}$

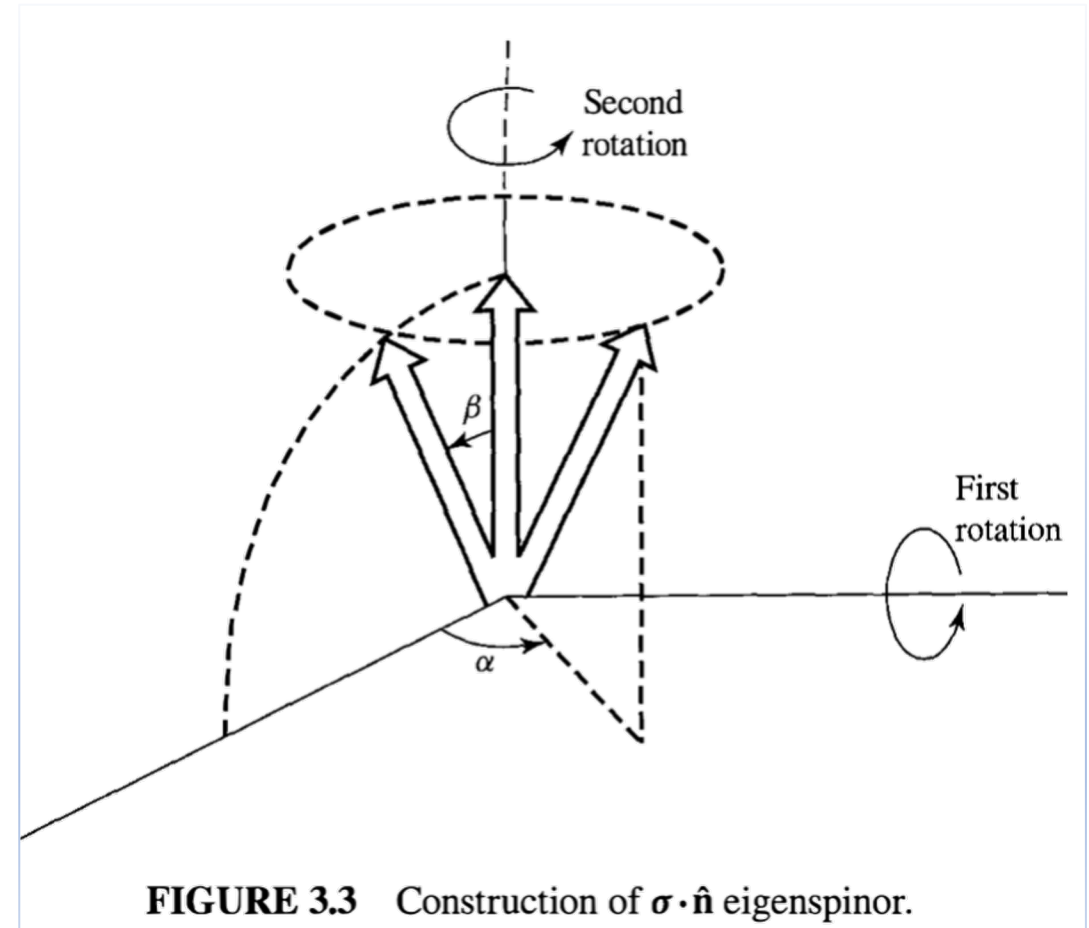
$$\sigma \cdot \hat{n} \chi = \chi$$

$\hat{n} \rightarrow$ unit vector,
polar angle: β ; azimuthal angle: α

$$\sigma = (\sigma_x, \sigma_y, \sigma_z)$$

$$\hat{n} = (\cos\beta\cos\alpha, \cos\beta\sin\alpha, \sin\beta)$$

$$\chi = \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) e^{-i\alpha/2} \\ \sin\left(\frac{\beta}{2}\right) e^{i\alpha/2} \end{pmatrix}$$



For spin-1/2 system

$$\sigma = \frac{2S}{\hbar}$$

$$\hat{n} = (0,0,1)$$

$$\sigma \cdot \hat{n} = \sigma_z$$

The first rotation, the operator : $e^{-iS_y\beta/\hbar}$

$|+\rangle$: eigenvector of σ_z

1: eigenvalue

The second rotation, the operator : $e^{-iS_z\alpha/\hbar}$

$$e^{-iS_z\alpha/\hbar} e^{-iS_y\beta/\hbar} |+\rangle = e^{-i\sigma_z\alpha/2} e^{-i\sigma_y\beta/2} |+\rangle$$

$$= \left[\cos \frac{\alpha}{2} - i \sigma_z \sin \frac{\alpha}{2} \right] \left[\cos \beta - i \sigma_y \sin \frac{\beta}{2} \right] \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) e^{-i\alpha/2} \\ \sin\left(\frac{\beta}{2}\right) e^{-i\alpha/2} \end{pmatrix}$$

Time reversal for a spin $\frac{1}{2}$ system

- $|\hat{n}; +\rangle = e^{-is_z\alpha/\hbar} e^{-is_y\beta/\hbar} |+\rangle$
- $\Theta |\hat{n}; +\rangle = e^{-is_z\alpha/\hbar} e^{-is_y\beta/\hbar} \Theta |+\rangle = \eta |\hat{n}; -\rangle$ (*)
- $|+\rangle$ is the initial state, the factor $e^{-is_z\alpha/\hbar} e^{-is_y\beta/\hbar}$ is the transformation of $|+\rangle$, $\Theta s \Theta^{-1} = -s$, $\Theta i \Theta^{-1} = -i$, then Θ directly operates on $|+\rangle$, it is the requirement for TRS, we expect $|\hat{n}; -\rangle$ (compare to the argument in P271, (4.4.22-4.4.24)),

η stands for an arbitrary

phase (a complex number of modulus unity)

$$|\hat{n}; -\rangle = e^{-is_z\alpha/\hbar} e^{-is_y(\pi+\beta)/\hbar} |+\rangle$$

$$\Theta = UK$$

Time reversal for a spin $\frac{1}{2}$ system

$$\Theta |\hat{n}; +\rangle = e^{-is_z\alpha/\hbar} e^{-is_y\beta/\hbar} \Theta |+\rangle = \eta |\hat{n}; -\rangle$$

$$\eta |\hat{n}; -\rangle = \eta e^{-is_z\alpha/\hbar} e^{-is_y(\pi+\beta)/\hbar} |+\rangle$$

$$\Theta = UK$$

$$\therefore U = \eta e^{-is_y\pi/\hbar}, K |+\rangle = |+\rangle$$

$|+\rangle$ is a real basis vector.

Identity

$$\begin{aligned} \bullet \exp\left(-\frac{i\vec{\sigma}\cdot\vec{n}\alpha}{2}\right) &= \left[1 - \frac{(\vec{\sigma}\cdot\vec{n})^2}{2!} \left(\frac{\alpha}{2}\right)^2 + \frac{(\vec{\sigma}\cdot\vec{n})^4}{4!} \left(\frac{\alpha}{2}\right)^4 - \dots \right] \\ &\quad - i \left[(\vec{\sigma}\cdot\vec{n}) \frac{\alpha}{2} - \frac{(\vec{\sigma}\cdot\vec{n})^3}{3!} \left(\frac{\alpha}{2}\right)^3 + \dots \right] \\ &= \mathbf{1} \cos\left(\frac{\alpha}{2}\right) - i (\vec{\sigma}\cdot\vec{n}) \sin\left(\frac{\alpha}{2}\right) \end{aligned}$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i \vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

$$(\vec{\sigma}\cdot\vec{\hat{n}})^n = \begin{cases} 1 & \text{for } n \text{ even} \\ \vec{\sigma}\cdot\vec{\hat{n}} & \text{for } n \text{ odd} \end{cases}$$

Time reversal for a spin $\frac{1}{2}$ system

$$\Theta = \eta e^{-i\pi S_y/\hbar} K = -i \eta \frac{2S_y}{\hbar} K = -i \eta \sigma_y K$$

Then we can check

$$\Theta |\hat{n}; +\rangle = e^{-is_z\alpha/\hbar} e^{-is_y\beta/\hbar} \Theta |+\rangle$$

$$\Theta e^{-is_z\alpha/\hbar} = e^{-is_z\alpha/\hbar} \Theta \quad (1)$$

$$\Theta e^{-is_y\beta/\hbar} = e^{-is_y\beta/\hbar} \Theta \quad (2)$$

In (1): $i \rightarrow -i$, $S_y S_z = -S_z S_y$

In (2): exponential factor is real. And S_y commutes with itself

Physical sense: $S \rightarrow -S$, similar to $J \rightarrow -J$

- $\mathcal{X}(\hat{n}; +) \rightarrow |\hat{n}; +\rangle$
- $\sigma \cdot \hat{n} \mathcal{X}(\hat{n}; +) = \mathcal{X}(\hat{n}; +)$, then
- $-i\sigma_y \mathcal{X}^*(\hat{n}; +) \rightarrow |\hat{n}; -\rangle$
- Choosing S_z representation

$$e^{-i\pi S_y/\hbar} |+\rangle = +|-\rangle, \quad e^{-i\pi S_y/\hbar} |-\rangle = -|+\rangle$$

$$\begin{aligned} \Theta (c_+ |+\rangle + c_- |-\rangle) &= +\eta c_+^* |-\rangle - \eta c_-^* |+\rangle, \\ \Theta^2 (c_+ |+\rangle + c_- |-\rangle) &= -|\eta|^2 c_+ |+\rangle - |\eta|^2 c_- |-\rangle \\ &= - (c_+ |+\rangle + c_- |-\rangle) \end{aligned}$$

$$\Theta^2 = -1 \rightarrow \text{kramers degeneracy}$$

证明:

$\because [\Theta, H] = 0, \quad H\psi = E\psi, H\Theta\psi = E\Theta\psi,$
 $\Theta\psi$ 与 ψ 对应同样的能量 E 。

假设 $\Theta\psi = e^{i\alpha}\psi$ (即两个波函数等价), 则有

$$\Theta^2\psi = \Theta e^{i\alpha}\psi = e^{-i\alpha}\Theta\psi = \psi \Rightarrow \Theta^2 = 1$$

与 $\Theta^2 = -1$ 相矛盾, 故波函数不等价, 二重简并。

Angular-momentum eigenstates:

- $\Theta^2 |j \text{ half - integer}\rangle = -|j \text{ half - integer}\rangle$ (1)
- $\Theta^2 |j \text{ integer}\rangle = +|j \text{ integer}\rangle$

The eigenvalue of Θ^2 is given by $(-1)^{2j}$

$$\Theta = \eta e^{-i\pi J_y/\hbar} K$$

For spherical harmonics

- $\Theta |l, m\rangle = (-1)^m |l, -m\rangle$
- Generally, $\Theta |j, m\rangle = (-1)^m |j, -m\rangle$ (j an integer), compatible with (1)

choose $\eta = i$, $\Theta |j, m\rangle = (i)^{2m} |j, -m\rangle$ for any j (half-integer or integer)

As a comparison, for the parity

$$P |l, m\rangle = (-1)^m |l, m\rangle$$

The presence of K (complex conjugate) makes Θ “anti-linear unitary.”

- Let $\Theta = UK$ $U^{-1}iU = i$, U is the unitary operator.

(1) for a spinless particle (U : phase factor)

$$\Theta = K \Rightarrow \Theta^2 = 1$$

(2) odd number of electrons

$$\Theta = -i\sigma_2 K \Rightarrow \Theta^2 = -1$$



Kramers degeneracy

Schur's lemma

Issai schur (1875-1941)

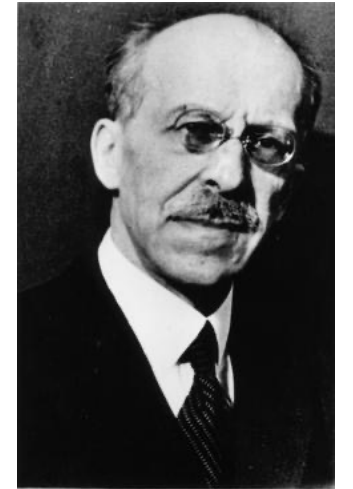
If $D(g)$ is an irreducible representation of a finite group G and if there is some matrix A such that $AD(g) = D(g)A$ for all g , then

$A = \lambda I$ for some constant λ .

What does it mean?

Mystery : degree of degeneracy = dimension of the irreducible representation

Span space : $\{G\psi\}$



I. Schur



Kramers' theorem

which states that the energy levels of a time-reversal invariant system with an odd number of electrons are n -fold degenerate where n is even(偶数).

- Essentially the energy levels come in pairs of Kramers doublets, and you can only split these pairs by **Hendrik Kramers** (1894–1952) introducing a perturbation that breaks time-reversal, such as a magnetic field.



我们回看一下实空间和 Hilbert 空间的“原路返回”的阐释，可以归结为 Wigner 在其 1959 年的著书[3]里对时间反演对称性的纲领性阐释：

一个系统在连续4 步操作后应该回到原来的状态，这 4 步操作依次为：

- 1, 时间反演;
- 2, 时间平移 Δt ;
- 3, 时间反演;
- 4, 时间平移 Δt 。

两次时间反演，就速度方向而言，相互补偿。形式表达为：

- (时间平移 Δt) \times (时间反演) \times (时间平移 Δt) \times (时间反演)= 单位算符

- 时间反演的灵魂是什么？

数学操作： $t \rightarrow -t$ ， 你希望什么保持不变？ 薛定谔方程？

Ok， 如果是， 需要取复共轭，
这样产生了所谓的反线性么正算符。

定义了state $|p, s\rangle$ ， 要求 $\Theta|p, s\rangle = |-p, -s\rangle$ ， 一切可以推导
从运动轨迹的反演 vs 态演化“轨迹”的反演

Up to a phase factor

Generalization : Poincaré group

- 保持Lorentz symmetry (time-space interval invariant) 的时空变换:
连续变换: 10个generators (Nöther theorem)。
离散变换: 空间反演, 时间反演

$$P = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad T = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ (之前的notation是}\theta\text{)}$$

在保证群乘法的意义上, 得到

$$P(iH)P^{-1} = iH, \quad T(iH)T^{-1} = -iH$$

$$p^u = (H, \vec{p})$$



1854 – 1912

Évariste Galois 1811-1832



Nous sommes des enfants, mais
les enfants bien courageux et
pleins en forme.
我们是孩子，但我们精力充沛，
勇往直前。。。

Emmy Nöther (1882 – 1935)



“我的方法是用抽象把问题剥离到本质”

“这不是我发明的，
这是数学本身告诉我们的。”

“mathematics is a wonderful gift we neither understand nor deserve.”

----- Eugene Wigner, who won the Nobel Prize for introducing group theory into quantum mechanics...



Eugene Wigner (1902 – 1995)

Detailed calculation: in the spin-less case

Variance or constant ? $t, \Delta t$

$$\begin{aligned} \Theta \frac{\partial}{\partial t} \psi &= \Theta \frac{\psi(t + \Delta t) - \psi(t)}{\Delta t} = \frac{\psi^*(-t + \Delta t) - \psi^*(-t)}{\Delta t} \\ &= - \frac{\psi^*(-(t - \Delta t)) - \psi^*(-t)}{-\Delta t} = - \frac{\partial \psi^*(-t)}{\partial t} = - \frac{\partial}{\partial t} \Theta \psi \end{aligned}$$

$$\Theta i = -i\Theta, \quad \Theta \frac{\partial}{\partial t} = - \frac{\partial}{\partial t} \Theta \quad \rightarrow \quad \left[\Theta, i \frac{\partial}{\partial t} \right] = 0$$

$$\Theta \psi(\vec{r}, t) = \psi^*(\vec{r}, -t),$$

$$\Theta e^{iHt} |\alpha\rangle = e^{iHt} \Theta |\alpha\rangle \quad (\text{here, } H\Theta = \Theta H)$$