Quasiparticle tunneling in the Moore-Read fractional quantum Hall state

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In fractional quantum Hall systems, quasiparticles of fractional charge can tunnel between the edges at a quantum point contact. Such tunneling (or backscattering) processes contribute to charge transport and provide information on both the charge and statistics of the quasiparticles involved. Here, we study quasiparticle tunneling in the Moore-Read state, in which quasiparticles of charges e/4 (non-Abelian) and e/2 (Abelian) may coexist and both contribute to edge transport. On a disk geometry, we calculate the matrix elements for e/2 and e/4 quasiholes to tunnel through the bulk of the Moore-Read state, in an attempt to understand their relative importance. We find that the tunneling amplitude for charge e/2 quasihole is exponentially smaller than that for charge e/4 quasihole, and the ratio between them can be (partially) attributed to their charge difference. We find that including long-range Coulomb interaction only has a weak effect on the ratio. We discuss briefly the relevance of these results to recent tunneling and interferometry experiments at filling factor \( \nu = 5/2 \).

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I. INTRODUCTION

The fractional quantum Hall effect (FQHE) at filling factor \( \nu = 5/2 \) (Refs. 1–10) has attracted strong interest due to the possibility that it may support non-Abelian quasiparticles and their potential application in topological quantum computation.11–14 Numerical studies15–22 indicate that the Moore-Read state23 or its particle-hole conjugate state24,25 is the most likely candidate to describe the \( \nu = 5/2 \) quantum Hall liquid. They both support non-Abelian quasiparticle excitations with fractional charge e/4, in addition to Abelian quasiparticle excitations with fractional charge e/2 of the Laughlin type.23,26

Edge excitations in the FQHE can be described at low energies by a chiral Luttinger liquid model27 and quasiparticle tunneling through barriers or constrictions was originally considered28,29 in the case of the Laughlin state. Recently the transport properties of the \( \nu = 5/2 \) state through a point contact have also been considered by a number of authors.30–32 Experimentally the quasiparticle charge of e/4 has been measured in the shot noise33 and temperature dependence of tunneling conductance.34 The latter also probes the tunneling exponent, which is related to the Abelian or non-Abelian nature of the state, although a direct probe on the statistics based on quasiparticle interference is desired.

The two-point-contact Fabry-Pérot interferometer was first proposed for probing the Abelian statistics35 and later considered for the non-Abelian statistics.36–45 In this kind of setup, quasiparticles propagating along the edges of the sample can tunnel from one edge to the other at the constrictions formed in a gated Hall bar. Such tunneling processes lead to interference of the edge current between two different tunneling trajectories. It has been used in both integer46,47 and fractional quantum Hall regimes in the lowest Landau level (LLL).48,49 Recently, Willett et al.50,51 implemented such a setup in the first excited Landau level (1LL) and attempted to probe the non-Abelian statistics of the quasiparticles in the case of \( \nu = 5/2 \) from the interference pattern.

The interference pattern at \( \nu = 5/2 \) state is predicted to exhibit an even-odd variation38,39 depending on the parity of the number of e/4 quasiparticles in the bulk. This would be a direct indication of their non-Abelian nature. In their experiments, Willett et al.51 observed oscillations of the longitudinal resistance while varying the side gate voltage in their interferometer. At low temperatures they observed apparent Aharonov-Bohm oscillation periods corresponding to e/4 quasiparticle tunneling for certain gate voltages and periods corresponding to e/2 quasiparticle at other gate voltages. This alternation was argued to be due to the non-Abelian nature of the e/4 quasiparticles,51,52 consistent with earlier theoretical prediction.38,39 At higher temperatures e/4 periods disappear while e/2 periods persist.50

There are two possible origins for the e/2 period in the interference picture: it may come from the interference of e/2 quasiparticles or from the interference of e/4 quasiparticles that traverse two laps around the interferometer. It is natural to expect that the tunneling of the e/4 quasiparticles is much easier than that of e/2 quasiparticles. Therefore, the tunneling amplitudes of e/4 quasiparticles should be larger than that of the e/2 quasiparticles. On the other hand, e/2 quasiparticles, being Abelian (or Laughlin type), involve the charge sector only and have much longer coherence length than that of e/4 quasiparticles.18 In fact it was predicted that the e/2 interference pattern will dominate once the temperature-dependent coherence length for e/4 quasiparticles becomes shorter than the distance between the two point contacts, in agreement with the recent experiment.50

In the present paper, we attempt to shed light on the relative importance of e/4 and e/2 quasiparticle tunneling in transport experiments involving point contacts. By numerically diagonalizing a special Hamiltonian with three-body
three-body interaction

The three-body interaction also generates a series of zero-energy states with a higher total angular momentum, related to edge excitations and bulk quasihole excitations. The N-electron Pfaffian state proposed by Moore and Read in the LLL representation,

$$\Psi_{MR}(z_1, z_2, \ldots, z_N) = \text{Pf} \left( \frac{1}{z_i - z_j} \prod_{i<j} (z_i - z_j)^2 \exp \left\{ -\sum_i |z_i|^2 \right\} \right),$$

is the exact zero-energy ground state of $H_{3B}$ with the smallest total angular momentum $M_0 = N(2N-3)/2$. In Eq. (2), the Pfaffian is defined by

$$\text{Pf} M_{ij} = \frac{1}{2^{N/2} (N/2)!} \sum_{\sigma \in S_N} \text{sgn} \sigma \prod_{k=1}^{N/2} M_{\sigma(2k-1)\sigma(2k)}$$

for an $N \times N$ antisymmetric matrix with elements $M_{ij}$.

The three-body interaction also generates a series of zero-energy states with a higher total angular momentum, related to edge excitations and bulk quasihole excitations. The N-electron Moore-Read ground state with an additional charge $e/4$ quasihole at the origin (so the edge also expands correspondingly due to a fixed number of electrons) has a wave function,

$$\Psi^{e/4}_{MR}(z_1, z_2, \ldots, z_N) = \text{Pf} \left( \frac{z_i + z_j}{z_i - z_j} \prod_{i<j} (z_i - z_j)^2 \exp \left\{ -\sum_i |z_i|^2 \right\} \right).$$

This state is a zero-energy state with total angular momentum $M_0 + N/2$ in the lowest $2N-1$ orbitals (one more than needed for the Moore-Read state), but not the only one. To generate the unique charge $e/4$ state, we need to introduce a strong repulsive interaction for electrons occupying the lowest two orbitals,

$$\Delta H_{e/4} = \kappa e^2 \sum_{ij} c_i^\dagger c_j^\dagger c_i c_j, \quad \kappa \to \infty.$$ 

On the other hand, the Moore-Read ground state with a $e/2$ quasihole (i.e., a Laughlin quasihole, equivalent to two $e/4$ quasiholes fused in the identity channel) at the origin,

$$\Psi^{e/2}_{MR}(z_1, z_2, \ldots, z_N) = \left( \prod_i z_i \right) \Psi_{MR}(z_1, z_2, \ldots, z_N),$$

is the unique zero-energy ground state with total angular momentum $M_0 + N$ in the lowest $2N-1$ orbitals. The Moore-Read state [Eq. (2)], together with its quasihole states [Eqs. (4) and (6)], can therefore be generated by numerically diagonalizing the three-body Hamiltonian [Eq. (1)] with Eq. (5), the special repulsion to generate an $e/4$ quasihole in the corresponding finite number of orbitals using the Lanczos algorithm. The wave functions can then be supplied to calculate the tunneling amplitudes for the quasiholes. The same numerical procedure can be used to study the tunneling amplitudes for the more realistic situation with a long-range interaction, in which case the variational wave functions are no longer eigenstates of the realistic Hamiltonian. For clarity and convenience, we will delay the discussion on how to generate realistic ground state and quasi-
hole states in the presence of long-range interaction (and their comparison with the variational states) to Sec. III B.

To study the tunneling amplitudes of the quasiholes, let us first consider a single-particle picture, which will help us understand our approach and, later, our results as well. In the disk geometry, the single-particle eigenstates are

$$|m\rangle = \phi_m(z) = (2\pi e^2 m!)^{-1/2} z^m e^{-|z|^2/4}. \quad (7)$$

We assume a single-particle tunneling potential

$$V_{\text{tunnel}}(\theta) = V_t(\theta), \quad (8)$$

which breaks the rotational symmetry. Here, we calculate the matrix element of $\langle k|V_{\text{tunnel}}(\theta)|l\rangle$, related to the tunneling of an electron from state $|l\rangle$ to state $|k\rangle$. One can visualize the tunneling process as a path along the polar angle $\theta=0$ between the two states centered on their maximum amplitudes at $|z| = \sqrt{2l}$ and $|z| = \sqrt{2k}$, respectively. One readily obtains

$$v_p(k, l) = \langle k|V_{\text{tunnel}}(\theta)|l\rangle = \frac{V_t}{2\pi} \frac{(k + l) + 1}{\sqrt{k + l} \Gamma}.$$ \quad (9)

The interesting limit is that we let $k$ and $l$ tend to infinity, but keep the tunneling distance fixed at $d$, i.e., $|k - l| \sim \sqrt{2k(d/l_B)} \ll (k + l)$. Alternatively, we can understand $d$ through the angular momentum change $l_B^2|k - l|/R$, where $R \sim \sqrt{2k} l_B$ is the azimuthal size of the single-particle state with momentum $k$ (or in this limit). We can show that (see Appendix A), in this limit,

$$v_p(k, l) \sim \frac{V_t}{2\pi} e^{-d^2/(2l_B^2)} \sim \frac{V_t}{2\pi} e^{-d^2/(2l_B^2)}, \quad (10)$$

which reflects the overlap of the two Gaussians separated by a distance $d$.

For quasiparticle tunneling at filling fraction $\nu=5/2$, one should—in principle—use wave functions in the 1LL. Evaluating the tunneling matrix element in the 1LL, we obtain an additional prefactor, so

$$v_p^{1\text{LL}}(k, l) = \left[ 1 - \frac{(k - l)^2}{2l_B^2} \right] v_p(k, l). \quad (11)$$

The sign change in the prefactor at $d \sim l_B$ can, unfortunately, cause severe finite-size effects for the numerically accessible range. Nevertheless, in the thermodynamic limit, the prefactor can be approximated by $-(k - l)^2/(2k + l) \sim -d^2/(2l_B^2)$ and, therefore, the leading decaying behavior is essentially the same. So we will continue to work in the 1LL but expect that the leading scaling behavior is the same as in the 1LL.

In the many-body case, we write the tunneling operator as the sum of the single-particle operators,

$$T = \sum \sum V_{\text{tunnel}}(\theta_i) \sum \partial(\theta_i). \quad (12)$$

We are now ready to calculate the tunneling amplitudes $\Gamma^{e4} = \langle \Psi_{\text{MR}}^{e4} | T | \Psi_{\text{MR}}^{e4} \rangle$ and $\Gamma^{e2/3} = \langle \Psi_{\text{MR}}^{e2/3} | T | \Psi_{\text{MR}}^{e2/3} \rangle$ for $e/4$ and $e/2$ quasiholes, respectively. For convenience, we will set $V_t$=1 as the unit of the tunneling amplitudes in the following text and figures. As explained in Ref. 53, the matrix elements consist of contributions from the respective Slater-determinant components $|l_1, \ldots, l_N\rangle \in \Psi_{\text{MR}}^{e4}$ and $|k_1, \ldots, k_N\rangle \in \Psi_{\text{MR}}^{e2/3}$. Nonzero contributions enter only when $|l_1, \ldots, l_N\rangle$ and $|k_1, \ldots, k_N\rangle$ are identical except for a single pair $l$ and $k$ with angular momentum difference $k - l = N/2$ or $N$ for the quasihole with charge $e/4$ or $e/2$. For clarity, we also include a pedagogical illustration of the procedure for calculating the tunneling matrix elements in the smallest possible system of four electrons in Appendix B.

III. RESULTS

A. Short-range interaction

Systems of up to six electrons can be worked out pedagogically using MATHEMATICA as illustrated in Appendix B. For larger systems, we obtain the exact Moore-Read and quasihole wave functions by the exact diagonalization of the three-body Hamiltonian (Eq. (11)) using the Lanczos algorithm. The tunneling amplitudes are then evaluated as explained in Sec. II. Figure 2(a) plots the tunneling amplitudes for the $e/4$ and $e/2$ quasiholes in the Moore-Read state as functions of electron number. The result for the $e/4$ quasihole shows a weak increase for $N=10$ followed by a decrease for $N>10$. On the other hand, the result for the $e/2$ quasihole shows a monotonic decrease as the number of electrons increases up to 14. In the largest system, the ratio of the two tunneling matrix elements is slightly less than 20. For comparison, we also plot the tunneling amplitudes $\Gamma^{e4}$ and $\Gamma^{e2/3}$ for the $e/3$ and $2e/3$ quasiholes in a Laughlin state at $\nu=1/3$ in Fig. 2(b). We also observe a bump in the tunneling amplitude $\Gamma^{e4}$ for charge $e/3$, followed by a monotonic decrease. We thus expect that $\Gamma^{e4}$ would eventually also show a monotonic decrease for large enough systems. $\Gamma^{e2/3}$ for charge $2e/3$ shows a much faster decrease, consistent with its larger charge, and thus a larger momentum transfer for the same tunneling distance.

Due to the finite-size bumps in $\Gamma^{e4}$ and $\Gamma^{e2/3}$ for charges $e/4$ and $e/3$, it is difficult to extract the asymptotic behavior in the tunneling amplitudes for these quasiparticles. However, we may expect that such finite-size corrections also
exist in the tunneling amplitude for charges $e/2$ and $2e/3$ so we can extract the asymptotic behavior in their ratios. Fortunately, this is indeed the case. We plot $\Gamma^{e/2}/\Gamma^{e/4}$ and $\Gamma^{2e/3}/\Gamma^{e/3}$ in Fig. 3. We find that the ratios can be fitted very well by exponentially decaying functions for almost all finite system sizes. The fitting results are
\begin{align}
\Gamma^{e/2}/\Gamma^{e/4} &\approx 1.78e^{-0.25N}, \\
\Gamma^{2e/3}/\Gamma^{e/3} &\approx 0.53e^{-0.40N}.
\end{align}

As will be discussed later, the exponents are related to the charge of the quasiholes and, to a lesser extent, to corrections due to sample geometry, perhaps also to the influence of the neutral component of the charge $e/4$ quasiparticles. Quantitatively, the constant in the exponent of the ratio $\Gamma^{e/2}/\Gamma^{e/4}$ is found to be smaller than that for $\Gamma^{2e/3}/\Gamma^{e/3}$, consistent with the smaller charge and thus smaller charge difference in the half-filled case.

One may question whether the tunneling amplitude for a quasihole from the disk center to the disk edge may be different from that for edge to edge, as in the realistic experimental situations. In particular, the former can contain a geometric factor, which can be corrected by mapping the disk to an annulus (or a ribbon) by inserting a large number of quasiholes at the disk center, from which electrons are repelled (see Appendix C for technical details). Inserting $n$ quasiholes to the center of a disk of $N$ electrons in the Moore-Read state, we can write the new wave function as
\begin{equation}
\Psi_{\text{MR}}^{e/2} = \left( \prod_{j=1}^{N} \psi_{\gamma_j}^{n} \right) \Psi_{\text{MR}},
\end{equation}
so that each component Slater determinant gets shifted into a new one to be normalized. The first $n$ orbitals from the center are now completely empty and the electrons are occupying orbitals from $n$ to $n+2N-3$. This transformation, of course, also changes the tunneling distance to
\begin{equation}
d(n,N)/l_B = \sqrt{2(n+2N-2)} - \sqrt{2n}.
\end{equation}

So we can plot data using $d(n,N)$, rather than $n$. Similarly, we can make the same transformation for the Moore-Read state with either an additional charge $e/4$ excitation or an additional charge $e/2$ excitation at the inner edge defined by the inserted $n$ quasiholes. Thus, we can calculate the tunneling amplitudes under the mapping from disk to annulus.

In Fig. 4, we show the tunneling amplitudes $\Gamma^{e/2}$ and $\Gamma^{e/4}$ for up to $n=100$ quasiholes. We plot them as functions of the tunneling distance $d$, which decreases as $n$ increases. It is interesting to note that finite-size effects diminish beyond $d>6l_B$ for charge $e/4$ and $d>5l_B$ for charge $e/2$. For comparison, we plot the ratio $\Gamma^{e/2}/\Gamma^{e/4}$ as a function of $d$ in Fig. 5. We find that, when we insert more than one quasihole, the ratio of the tunneling amplitudes falls onto a single curve, regardless of the system size $N$ and the number of quasiholes $n$. The curve can be fit roughly to
\begin{equation}
\frac{\Gamma^{e/2}(d)}{\Gamma^{e/4}(d)} \approx e^{-0.083(d/l_B)^2}.
\end{equation}

We point out that a few points in Fig. 5 can be seen deviating from this behavior. They correspond to the largest $d$ for a given $N$, meaning that there is no quasihole in the bulk, thus...
corresponding to the bulk-to-edge instead of the edge-to-edge tunneling.

It is worth pointing out that such a behavior is not completely unexpected; in fact, it reflects the asymptotic behavior of the single-particle tunneling matrix elements and the corresponding charge of the quasiparticles. To see this, we note that for a charge $q$ quasihole to tunnel a distance of $d$, one electron (in each Slater determinant) must hop by a distance of $qd/e$ for the exact momentum transfer. According to the asymptotic behavior in Eq. (10), we expect

$$\Gamma_d \sim e^{-(qd/2e)^2}. \quad (18)$$

Therefore, we expect

$$\Gamma_d^{1/2} \sim e^{-(d/2)^2-(d/4)^2/2}\Gamma_d^{1/4} \approx e^{-0.047(d/4)^2}, \quad (19)$$

which we also include in Fig. 5 for comparison.

We thus find that both variational wave function calculation and qualitative analysis suggest that the tunneling amplitude of the $e/2$ quasiparticles is smaller than that of the $e/4$ quasiparticles by a Gaussian factor in edge-to-edge distance $d$, which is the main results of this paper. There is, however, a quantitatively discrepancy in the length scale associated with the Gaussian dependence between Eqs. (17) and (19). This indicates that the Gaussian factor in single-electron tunneling matrix element only partially accounts for the Gaussian dependence; the remaining decaying factor thus must be of many-body origin, whose nature is not clear at present and warrants further study.

**B. Long-range interaction**

So far, we have discussed the tunneling amplitudes using the variational wave functions, which are exact ground states of the three-body Hamiltonian. These wave functions are unique, but in general not the exact ground states of any generic Hamiltonian one may encounter in a realistic sample. In reality, long-range Coulomb interaction is overwhelming, although Landau-level mixing can generate effective three-body interaction.54 In this subsection, we explore the quasihole tunneling in the presence of long-range Coulomb interaction. The central questions are the following. First, how can we generate both non-Abelian and Abelian quasiholes in practice? Remember that now we do not have the variational Moore-Read state as the exact ground state, so the variational quasihole states may also be less meaningful. We attempt to generate and localize quasiholes with a single-body impurity potential; then, how close are the corresponding wave functions to the variational ones? Second, suppose we have well-defined quasihole wave functions; are the results on the tunneling amplitude obtained in the short-range three-body interaction case robust in the presence of long-range Coulomb interaction?

For a smooth interpolation between the short- and long-range cases, we introduce a mixed Hamiltonian,

$$H_\lambda = (1 - \lambda)H_C + \lambda H_{3B}, \quad (20)$$

as explained in our earlier works.17,18 Here, the dimensionless $\lambda$ interpolates smoothly between the limiting cases of the three-body Hamiltonian $H_{3B} (\lambda = 1)$ and a two-body Coulomb Hamiltonian $H_C (\lambda = 0)$. $H_C$ also includes a background confining potential arising from neutralizing background charge distributed uniformly on a parallel disk of radius $R = \sqrt{4N}$, located at a distance $D$ above the two-dimensional electron gas. Using the symmetric gauge, we can write down the Hamiltonian for electrons in the 1LL as

$$H_C = \frac{1}{2} \sum_{mnl} V_{mn}^c c^\dagger_m c^\dagger_n c^\dagger_l + \sum_m U_m c^\dagger_m c_m, \quad (21)$$

where $c^\dagger_m$ is the electron creation operator for the 1LL single-electron state with angular momentum $m$. $V_{mn}^c$'s are the corresponding matrix elements of Coulomb interaction for the symmetric gauge and $U_m$'s the corresponding matrix elements of the confining potential. We fix $D = 0.6 l_B$ so the ground state can be well described by the Moore-Read state.

To be experimentally relevant, we also want to generate the quasihole states by a generic impurity potential, rather than by the special interaction [Eq. (5)] we used above to generate the unique $e/4$ quasihole state in the three-body case. We consider a Gaussian impurity potential,55

$$H_{imp}(W, s) = W \sum_m e^{-m^2/2\sigma^2} c^\dagger_m c_m, \quad (22)$$

which will trap at the disk center an $e/4$ or $e/2$ quasihole depending on its strength.18 Here, $s$ characterizes the range of the potential. Note that $H_{imp} = W c_0^\dagger c_0$ is the short-range limit ($s \to 0$) of the Gaussian potential in Eq. (22). $W$ is always expressed in units of $e^2/(\ell_B^2)$.

Earlier studies18,55 have identified $s = 2.0$ as a suitable width for the Gaussian trapping potential, which is of roughly the radial size of a quasihole. So we use this value exclusively in the following discussion. One expects that, for small $W$, the system remains in the Moore-Read phase without any quasihole excitation in the bulk; for later reference, we use $E_0^\lambda$ to denote the ground-state energy in the momentum subspace of $M = M_0 + N(2N-3)/2$. As $W$ increases, the impurity potential first tends to attract a charge $e/4$ quasihole, the smallest charge excitation, at the disk center. This would be reflected in the sudden angular momentum change from $M_0$ to $M_0 + N/2$ of the global ground state, which is also characterized by a depletion of $1/4$ of an electron in the electron occupation number at orbitals with small momentum. We use $E_\lambda$ to denote the ground-state energy in the subspace of $M = M_0 + N/2$. When $W$ is increased further, one can trap a charge $e/2$ quasihole at the center, with ground state having the total angular momentum of $M_0 + N/2$ in this momentum subspace, we use $E_\lambda$ to denote the ground-state energy. We illustrate this scenario for a 12-electron system with $\lambda = 0.5$ in Fig. 6(a), in which we plot the energies of the $e/4$ and $e/2$ quasihole states $E_\lambda^{e/4}$ and $E_\lambda^{e/2}$, measured from the corresponding $E_\lambda^{e/4}$. More precisely, the $e/4$ quasihole state is energetically favorable for $0.032 < W < 0.137$. At $W < W_c^{e/4} = 0.032$, we find $E_\lambda^{e/4} < E_\lambda^{e/2}$, while at $W > W_c^{e/2} = 0.137$, we find $E_\lambda^{e/4} > E_\lambda^{e/2}$.

To understand how good these wave functions are, we plot in Fig. 6(b) the overlap of the $e/4$ quasihole state $|\Psi_\lambda^{e/4}\rangle$ with the corresponding variational state $|\Psi_{MR}^{e/4}\rangle$ [Eq. (4)], as
Read phase. The fact that CHEN et al. has no simultaneous occupation of the lowest two orbitals.

We induce charge see the excellent agreement between states \( \equiv 0.1 \), at which we have times that are rather robust. In particular, we choose Coulomb interaction.

Laps decrease as we increase the percentage of the long-range Moore-Read state, although we already mix in a considerable surprisingly high overlaps of the quasihole wave functions to well-separated localized non-Abelian anyons. Second, the notion of topological quantum computation, for which we need charge

Let us make some comments here. First, as we tune up

Therefore, we expect that, with a moderate mixture of the long-range Coulomb interaction, the results on the tunneling amplitudes are rather robust. In particular, we choose \( W = 0.1 \), at which we have \( E^0_\alpha > E^{e/2}_\alpha > E^{e/4}_\alpha \) and at which both \( |\langle \Psi^e_\alpha | \Psi^{e/4}_\alpha \rangle| \) and \( |\langle \Psi^e_\alpha | \Psi^{e/2}_\alpha \rangle| \) are very close to 1. For example, we plot the ratio of tunneling amplitudes \( \Gamma^{e/2}/\Gamma^{e/4} \) as a function of the number of electrons for \( \lambda = 0.5 \) in Fig. 7.

The data points are in good agreement with the trend [Eq. (13)] obtained earlier for the pure three-body case.

So far, we have shown a case where the presence of the long-range interaction has very weak effects on the results of tunneling amplitudes. However, in general, one can expect that such an agreement becomes worse as one moves farther away from the pure repulsive three-body interaction in the parameter space. To present a more quantitative picture, we plot in Fig. 8 the ratio of tunneling amplitudes \( \Gamma^{e/2}/\Gamma^{e/4} \) (without inserting quasiholes at the center, i.e., \( n=0 \)) as a function of \( \lambda \) for the 12-electron system with the mixed Hamiltonian \( H_\lambda \) and a Gaussian trapping potential (\( W=2.0, s=1.0 \)). The ratio remains as a constant from \( \lambda = 1 \) down to 0.2 before it fluctuates significantly; the fluctuation is believed to be related to the stripelike phase near the pure Coulomb case in finite systems, as also revealed in an earlier work.18 We point out that recent numerical work suggests
that the spin-polarized Coulomb ground state at $\nu=5/2$ is
adiabatically connected with the Moore-Read wave function
for systems on the surface of a sphere,\(^{22}\) so the large
deviation may well be a finite-size artifact.

Varying parameters, such as $W$, $s$, and $D$, can also lead
to a larger deviation from the pure three-body case, although
we find that in generic cases $\Gamma^{e/2}/\Gamma^{e/4}$ remains small. We
remind the reader that the Moore-Read phase is extremely
fragile. Therefore, we have rather strong constraints on param-
eters when both the Moore-Read-like ground state and the
quasihole states should subsequently be a good description
of the ground states as the impurity potential strength
increases. For example, the window of $D$ for the ground state
at $W=0$ to be of Moore-Read nature in the pure Coulomb
case is very narrow ($0.51 < D/l_B < 0.76$ for 12 electrons in
22 orbits\(^{17}\)); in this range, the effect of the background
potential parameter $D$ on the ratio of tunneling amplitudes is
negligible (less than 1% variation). Therefore, we conclude
that the small ratio of $\Gamma^{e/2}/\Gamma^{e/4}$ is robust in the presence of
the long-range interaction as long as the system remains in
the Moore-Read phase.

IV. DISCUSSION

In this work we use a simple microscopic model to study
quasiparticle tunneling between two fractional quantum Hall
edges. We find that the tunneling amplitude ratio of quasi-
particles with different charges decays with a Gaussian tail as
the edge-to-edge distance increases. The characteristic length
scale associated with this dependence can be partially
accounted for by the difference in the charges of the corre-
sponding quasiparticles. More specifically, we find that the
tunneling amplitude for a charge $e/4$ quasiparticle is signifi-
cantly larger than that for a charge $e/2$ quasiparticle in the
Moore-Read quantum Hall state, which may describe the ob-
served fractional quantum Hall effect at the filling factor $\nu=
5/2$. This result was anticipated in Ref.\(^{52}\), in which the
authors outlined microscopic calculations that are similar to
the discussion in Sec. III A (see their Appendix B).

It is worth emphasizing that what we have calculated here
are the bare tunneling amplitudes. Under renormalization-
group (RG) transformations, both amplitudes will grow as
one goes to lower energy or temperature, as they are both
relevant couplings in the RG sense. The ratio between them,
$\Gamma^{e/2}/\Gamma^{e/4}$, will decrease under RG because $\Gamma^{e/4}$ is more
relevant than $\Gamma^{e/2}$, which renders $\Gamma^{e/2}$ even less important than
$\Gamma^{e/4}$ at low temperatures. This agrees with recent tunneling
experiments involving a single point contact,\(^{33,34}\) where the
best fits of the data are more consistent with charge $e/4$
quasiparticle tunneling than charge $e/2$ quasiparticle
tunneling.

However, the importance of the two kinds of quasiparti-
cles can be reversed in interferometry experiments that look
for signatures from interference between two point contacts.
This is because the interference signal depends not only on
the quasiparticle tunneling amplitudes, but also on their co-
herence lengths when propagating along the edge of frac-
tional quantum Hall samples. Recently, Vishara and Nayak\(^{41}\)
found that in a double-point-contact interferometer, the oscil-

FIG. 9. (Color online) Decoherence length $L_\phi$ as a function of $D$
for both $e/4$ (upper line) and $e/2$ (lower line) quasiparticles in
the Moore-Read Pfaffian state. We choose a temperature $T=25$ mK to
allow a direct comparison with experiment (Ref. 50). The broken
lines above $D=0.62l_B$ are obtained by extrapolation, as the Moore-
Read-like ground state is no longer stable in a system of 12 elec-
trons in 26 orbitals. We note that a stripe phase may emerge below
$D=1.2l_B$ (Ref. 18).

lating part of the current for charge $q$ quasiparticles can be written
as

$$I_{12}^{(q)} \propto \gamma \Gamma^{(q)} \left| \Gamma^{(q)} \right|^2 e^{-\xi_{12}/\xi_{12}^{(q)}} \cos \left( \frac{2\pi q\Phi}{e\phi_0} + \delta^{(q)} + \alpha \right),$$

(23)

where $\Gamma^{(q)}_{1/2}$ are the charge $q$ quasiparticle tunneling am-
plitudes at the two quantum point contacts 1 and 2 with a dis-
tance of $x_{12}$. $\gamma$ is a suppression factor resulting from the
possible non-Abelian statistics of the quasiparticles. For $q=
e/2$, we have $\gamma=1$, while for $q=\pm e/4$, $\gamma=\pm 1/\sqrt{2}$ (0) when
we have even (odd) number of $e/4$ quasiparticles in bulk.
The sign depends on whether the even number of $e/4$ qua-
siparticles fuses into the identity channel (+) or the fermionic
channel (−). $\Phi$ is the flux enclosed in the interference loop
and $\Phi_0=hc/e$ is the magnetic flux quantum. The phase $\delta^{(q)}$
is the statistical phase due to the existence of bulk quasiparti-
cles inside the loop and $\alpha$ is the phase arg($\Gamma^{(q)}\Gamma^{(q)}$). At a
finite temperature $T$, the decoherence length $L_{\phi}^{(q)}$ for the quasiparticle
in the Moore-Read state is\(^{41}\)

$$L_{\phi}^{(q)} = \frac{1}{2\pi T} \left( \frac{\delta^{(q)}_{\alpha}}{v_c} + \frac{g_{\alpha}^{(q)}}{v_n} \right)^{-1},$$

(24)

where $v_{c,n}$ are the charge and neutral edge mode velocities and
$g_{\alpha}^{(q)}$ are the charge and neutral sector scaling exponents
for charge $q$ quasiparticles, respectively. Earlier studies by
the authors\(^{17,18}\) found that the neutral velocity can be signifi-
cantly smaller (by a factor of 10) than the charge velocity,
leading to a shorter coherence length $L_{\phi}^{(e/4)}$ for charge $e/4$
quasiparticles (less than 1/3 of $L_{\phi}^{(e/2)}$ for charge $e/2$ quasiparticles
in the Moore-Read case, as $L_{\phi}^{(e/2)}$ depends on $v_{c}$ only
because it is Abelian and $g_{\alpha}^{(e/2)}=0$). Finite-size numerical
analysis\(^{56}\) maps out the dependence of $L_{\phi}^{(e/4)}$ and $L_{\phi}^{(e/2)}$ on
the strength of the confining potential parametrized by $D$ for
the Moore-Read state, as summarized in Fig. 9 for $T=25$ mK.
APPENDIX A: SINGLE-PARTICLE TUNNELING MATRIX ELEMENTS IN THE LARGE DISTANCE LIMIT

In the disk geometry, the single-particle eigenstates are

$$|m\rangle = \phi_m(e) = (2\pi)^{m/2}e^{-e^2/4} e^{-x^2/4}.$$  \hspace{1cm} (A1)

If we assume a single-particle tunneling potential

$$V_{\text{tunnel}} = V_r \delta(\theta),$$ \hspace{1cm} (A2)

the matrix element of $\langle k|V_t|l \rangle$, related to the tunneling of an electron from state $|l \rangle$ to state $|k \rangle$, is

$$v_p(k,l) = \langle k|V_{\text{tunnel}}|l \rangle = \frac{V_p}{2\pi} \frac{k+l+1}{k! l!}. \hspace{1cm} (A3)$$

Using beta functions

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \hspace{1cm} (A4)$$

we can rewrite the dimensionless tunneling matrix element as

$$\tilde{v}_p(k,l) = \frac{2\pi v_p(k,l)}{V_r} = \left[ B\left( \frac{k+l+1}{2}, \frac{k+l+1}{2} \right) \right]^{1/2}. \hspace{1cm} (A5)$$

We are interested in the limit of large $l$ and large $k$, where we can use the asymptotic formula of Stirling’s approximation,

$$B(x,y) \sim \sqrt{2\pi} \left( \frac{x^{-1/2}y^{-1/2}}{(x+y)^{1/2}} \right). \hspace{1cm} (A6)$$

for large $x$ and large $y$. Therefore, we have

$$\tilde{v}_p(k,l) \sim \left[ \frac{(k+l+1)^{k+l+1}}{(k+1)^{(k+1)^{k+1}}(l+1)^{l+1}} \right]^{1/2}. \hspace{1cm} (A7)$$

For convenience, we define

$$S = \frac{k+l}{2}, \quad M = \frac{k-l}{2}. \hspace{1cm} (A8)$$

If we further take the limit of $S \gg |M|$, we find

$$\tilde{v}_p(k,l) \sim \left[ 1 + \frac{M}{S+1} \right]^{-[(S+M)/2]-1/4} \left[ 1 + \frac{M}{S+1} \right]^{-[(S-M)/2]-1/4} \sim \left[ 1 + \frac{M}{S+1} \right]^{-M} \sim e^{-M^2[(S+1)/2]}.$$  \hspace{1cm} (A9)

APPENDIX B: TUNNELING MATRIX ELEMENTS IN FEW-ELECTRON SYSTEMS

In this appendix, we first illustrate the calculation of the tunneling matrix elements in a four-electron system for the
Moore-Read state. In this case, the normalized Moore-Read wave function can be written as a sum of Slater determinants as
\begin{equation}
\Psi_{\text{MR}} = \frac{1}{\sqrt{10}}[011110] - \frac{1}{2}[101101] + [110011],
\end{equation}
where the ket notation denotes a Slater determinant with electrons occupying the single-particle orbitals labeled by 1. For example, [110011] means the normalized antisymmetric wave function of four electrons occupying the orbitals with angular momenta of 0,1,5,6 (reading from left to right in the ket). This can be obtained by explicitly expanding the Moore-Read state with four electrons (trivial with the help of MATHEMATICA). The corresponding \( e/4 \) quasihole state, similarly, can be written as
\begin{equation}
\Psi_{\text{MR}}^{e/4} = \frac{1}{5\sqrt{11}}(\sqrt{3}[010101] - 4\sqrt{6}[0101110] + 8\sqrt{2}[0110110] - 4\sqrt{3}[0111001]),
\end{equation}
and the \( e/2 \) quasihole state,
\begin{equation}
\Psi_{\text{MR}}^{e/2} = \frac{1}{3\sqrt{13}}(10[0011110] - 2\sqrt{3}[0101101] + \sqrt{5}[0110011]).
\end{equation}

In the many-body case, we write the tunneling operator as the sum of the single-particle operators,
\begin{equation}
T = V_{\text{t}} \sum_{i=1}^{N} \delta(\theta_i),
\end{equation}
and calculate the tunneling amplitudes \( \Gamma^{e/4} = \langle \Psi_{\text{MR}} | T | \Psi_{\text{MR}}^{e/4} \rangle \) and \( \Gamma^{e/2} = \langle \Psi_{\text{MR}} | T | \Psi_{\text{MR}}^{e/2} \rangle \) for \( e/4 \) and \( e/2 \) quasiholes, respectively. The matrix elements consist of contributions from the respective Slater-determinant components \( |l_1, \ldots, l_N \rangle \in \Psi_{\text{MR}} \) and \( |k_1, \ldots, k_N \rangle \in \Psi_{\text{MR}}^{e/4} \) or \( \Psi_{\text{MR}}^{e/2} \). There are nonzero contributions only when the two sets \( |l_1, \ldots, l_N \rangle \) and \( |k_1, \ldots, k_N \rangle \) are identical except for a single pair \( \tilde{l} \) and \( \tilde{k} \) with angular momentum difference \( \tilde{k} - \tilde{l} = N/2 \) or \( N \) for the quasihole with charge \( e/4 \) or \( e/2 \). One should also pay proper attention to fermionic signs.

With some algebra, one obtains, for the four-electron case,
\begin{equation}
\langle \Psi_{\text{MR}} | T | \Psi_{\text{MR}}^{e/4} \rangle = \frac{1}{5\sqrt{13}} [16 \sqrt{2} v_{p}(3,5) + 4 \sqrt{3} v_{p}(4,6) + 8 \sqrt{3} v_{p}(2,4) + 8 \sqrt{2} v_{p}(0,2) + 4 \sqrt{2} v_{p}(1,3)],
\end{equation}
\begin{equation}
\langle \Psi_{\text{MR}} | T | \Psi_{\text{MR}}^{e/2} \rangle = \frac{1}{39} [10 \sqrt{10} v_{p}(1,5) + 10 \sqrt{2} v_{p}(0,4) + 2 \sqrt{3} v_{p}(2,6)],
\end{equation}
where, as before, we define
\begin{equation}
v_{p}(k,l) = V_{\text{t}} \left( \frac{k + l}{2} + 1 \right) \sqrt{|k-l|!}.
\end{equation}
The numerical values for the two tunneling matrix elements are 0.213 and 0.123, respectively, in units of \( V_{\text{t}} \). Therefore, in the smallest nontrivial system, we find that the tunneling amplitude for \( e/4 \) quasiholes is roughly twice as large as that for \( e/2 \) quasiholes. The example of the four-electron case illustrates how the tunneling amplitudes can be computed. The results are, however, not particularly meaningful as the system size is so small that one cannot really distinguish bulk from edge.

A similar analysis can be performed for a system of six electrons with the help of MATHEMATICA. Due to larger Hilbert space, we will not explicitly write down the decomposition of the ground states and quasihole states by Slater determinants. Instead, we only point out that the tunneling matrix elements are given by
\begin{equation}
\langle \Psi_{\text{MR}} | T | \Psi_{\text{MR}}^{e/4} \rangle = 0.267,
\end{equation}
\begin{equation}
\langle \Psi_{\text{MR}} | T | \Psi_{\text{MR}}^{e/2} \rangle = 0.105,
\end{equation}
in units of \( V_{\text{t}} \).

**APPENDIX C: MAPPING FROM DISK TO ANNULUS**

Microscopic quantum Hall calculations are commonly based on one of the following geometries (or topologies): torus, sphere, annulus (or cylinder), and disk. In a specific calculation, they are chosen either for convenience or for the need for having different numbers of edge(s). On the other hand one can also map one geometry to another by means of, e.g., quasihole insertion. Here, to connect the theoretical analysis with experiment, we perform a mapping from the disk to the annulus geometry by inserting a large number of quasiholes at the center of the disk, effectively creating an inner edge, as the electron density in the center is suppressed by inserting a small disk of Laughlin quasihole liquid.

After inserting \( n \) charge \( e/2 \) Laughlin quasiholes to the center of an \( N \)-electron Moore-Read state, the ground state can be written as
\begin{equation}
\Psi_{\text{MR}}^{n/2} = \prod_{i=1}^{N} \left( \sum_{e=1}^{N} \phi_{e} \right) \Psi_{\text{MR}},
\end{equation}
where the additional factor transforms each Slater determinant into a new one to be normalized. Let us use the case of four electrons as in Appendix B to illustrate. We note that a Slater determinant \( |011110\rangle \) with an addition of \( n \) Laughlin quasiholes evolves into another Slater determinant \( |0^{n}011110\rangle \), meaning that the \( m \)th (in this example, \( m = 1 - 4 \)) single-particle orbital is now mapped to the \( (m + n) \)th orbital. Due to the difference in normalization, the latter determinant should be multiplied by a factor of \( F(n; 1, 2, 3, 4) \) with a general form of
\[ F(n;m_1, m_2, \ldots, m_N) = 2^{nN/2} \prod_{i=1}^{N} \frac{(n + m_i)!}{m_i!}. \] (C2)

Therefore, when we express Eq. (C1) explicitly for Eq. (B1), we have

\[
\Psi_{\text{MR}}^{n/e/2} = \mathcal{N} \left[ F(n;1,2,3,4) \sqrt{\frac{10}{\sqrt{13}}} (0^n)011110 \right.
\]
\[- F(n;0,2,3,5) \sqrt{\frac{2}{\sqrt{13}}} (0^n)101101 \]
\[+ F(n;0,1,4,5) \frac{1}{\sqrt{13}} (0^n)110011 \right], \] (C3)

where \( \mathcal{N} \) is a numerical normalization factor. For \( n=1 \) we thus obtain exactly Eq. (B3) as expected. Interestingly, in the \( n \rightarrow \infty \) (ring) limit, the normalized wave function becomes, asymptotically,

\[
\Psi_{\text{MR}}^{n/e/2} = \mathcal{C} \left[ \sqrt{\frac{1}{12!3!4!\sqrt{13}}} (0^n)011110 \right.
\]
\[+ \sqrt{\frac{2}{0!1!2!3!5!\sqrt{13}}} (0^n)110011 \right], \] (C4)

where the normalization factor \( \mathcal{C} \) is, explicitly,

\[
\frac{1}{\mathcal{C}} = \sqrt{\frac{1}{12!3!4!13} \frac{1}{0!2!3!5!13} \frac{1}{0!1!4!5!13}}.
\] (C5)

In the limit of \( n \gg N \), we have \( v_J(n + m_1, n + m_2) \rightarrow v_J/2 \). Therefore, the tunneling matrix between the states with \( n \) and \( n+1 \) quasiholes becomes

\[
\langle \Psi_{\text{MR}}^{n/e/2} | T | \Psi_{\text{MR}}^{n/e/2+1/2} \rangle = V_T \mathcal{C}^2 \left[ \sqrt{\frac{1}{12!3!4!13}} + \sqrt{\frac{2}{0!1!2!3!5!13}} \right],
\] (C6)

The tunneling of \( e/4 \) quasiholes can be worked out in a similar fashion and we can obtain generically \( \langle \Psi_{\text{MR}}^{n/e/2} | T | \Psi_{\text{MR}}^{n/e/2+e/4} \rangle \).
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